Putnam Training
Number Theory

November 2, 2021

1. Let a, b, c be integers such that $a^2 + b^2 = c^2$. Prove that $3 \mid ab$.
2. Given natural numbers a, b, c such that $a+b+c$ is divisible by 6, prove that $a^3+b^3+c^3$ is also divisible by 6.
3. Show that among any three distinct integers, we can find two, say a and b, such that $a^3b - ab^3$ is divisible by 10.
4. Prove that there are no natural numbers a and b such that $a^2 - 3b^2 = 8$.
5. Given the pair of prime numbers p and p^2+2 show that p^3+2 is also a prime number.
6. If p, $4p^2+1$, and $6p^2+1$ are prime numbers, find p.
7. If $2n+1$ and $3n+1$ are both perfect squares, prove that n is divisible by 40.
8. Find the last digit of $2022^{2022^{2022}}$.
9. (USAMO, 1979) Find all natural number solutions $(n_1, n_2, ..., n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

10. (a) Let n be a natural number with digits $d_k, d_{k-1}, \ldots, d_1, d_0$. Show that

$$n \equiv d_k + d_{k-1} + \cdots + d_1 + d_0 \mod 9.$$

    (b) The number 2^{29} has 9 distinct digits. Without using a calculator, tell which digit is missing.

    (c) (IMO, 1975) Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

11. (a) Let p be a prime number. Prove that for every natural number a, there exists a natural b such that $p \mid ab - 1$.

    (b) Is the above statement true if p is not a prime number?

    (c) (Wilson's theorem) Prove that $(n-1)! + 1$ is divisible by n if and only if n is prime.

12. Prove that there exits a natural number n such that $n+1$, $n+2$, $\ldots$, $n+2020$ are all composite numbers.

Congruences and Modular Arithmetic

Congruences: We say a is congruent to b modulo m, and write

$$a \equiv b \mod m,$$

if a and b have the same remainder when divided by m. For example:

- $5 \equiv 17 \mod 3$,
- $10 \equiv -4 \mod 7$,
- $n \equiv d \mod 10$ if n has d as last decimal digit.

Modular Arithmetic: The key fact about congruences is that congruences to the same modulus can be added, multiplied, and taken to a fixed positive integral power. For example:

- $5 \cdot 10 + 0 \equiv 2 \cdot 1 + 0 \mod 3$,
- $2019^{2020} \equiv 9^{2020} \mod 10$.

Congruence and powers: A number when divided by n can only have reminders $0, 1, \ldots, n-1$. So powers of a fixed number modulo n will eventually repeat. For example:

Q. What is the last digit of 2019^{2020}?

- $2019^1 \equiv 9^1 \equiv 9 \mod 10$,
- $2019^2 \equiv 9^2 \equiv 1 \mod 10$,
- $2019^3 \equiv 9^3 \equiv 9 \mod 10$,
- $2019^4 \equiv 9^4 \equiv 1 \mod 10, \ldots$

From the above pattern $2019^{2020} \equiv 1 \mod 10$, so the last digit of 2019^{2020} is 1.

Exhausting cases: A number when divided by n can only have reminders $0, 1, \ldots, n-1$. It is often possible to test all the possible remainders and look for patterns. The challenge is trying to guess which n to use. For example:

Q. Show that 4000000007 is not a perfect square.

- $0^2 \equiv 0 \mod 4$,
- $1^2 \equiv 1 \mod 4$,
- $2^2 \equiv 0 \mod 4$,
- $3^2 \equiv 1 \mod 4$.

So squares modulo 4 are either 0 or 1. But $4000000007 \equiv 3 \mod 4$, so it is not a perfect square.

Good guesses for n are powers of a prime e.g. 2, 3, 4, 5, 8, 16, etc.