

# MATHEMATICAL INDUCTION

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(Last updated: October 12, 2015)

## MATHEMATICAL INDUCTION (SUMMARY)

This is a powerful method to prove properties of positive integers.

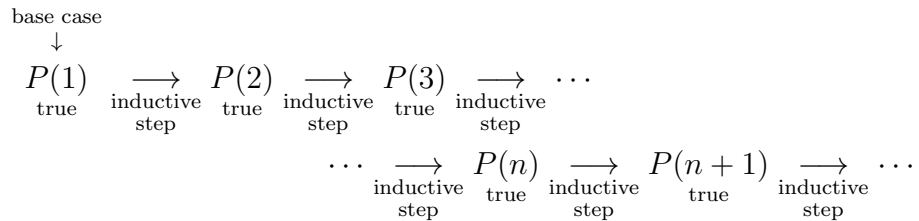
**1. Principle of Mathematical Induction.** Let  $P$  be a property of positive integers such that:

- (1) *Base Case:*  $P(1)$  is true, and
- (2) *Inductive Step:* if  $P(n)$  is true, then  $P(n + 1)$  is true.

Then  $P(n)$  is true for all positive integers.

The premise  $P(n)$  in the inductive step is called *Induction Hypothesis*.

The validity of the Principle of Mathematical Induction is obvious. The base case states that  $P(1)$  is true. Then the inductive step implies that  $P(2)$  is also true. By the inductive step again we see that  $P(3)$  is true, and so on. Consequently the property must be true for all positive integers:



*Example:* Prove that the sum of the  $n$  first odd positive integers is  $n^2$ , i.e.,  $1 + 3 + 5 + \dots + (2n - 1) = n^2$ .

*Answer:* Let  $S(n) = 1 + 3 + 5 + \dots + (2n - 1)$ . We want to prove by induction that for every positive integer  $n$ ,  $S(n) = n^2$ .

- (1) *Base Case*: If  $n = 1$  we have  $S(1) = 1 = 1^2$ , so the property is true for 1.
- (2) *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer  $n$ , i.e.:  $S(n) = n^2$ . We must prove that it is also true for  $n + 1$ , i.e.,  $S(n + 1) = (n + 1)^2$ . In fact:

$$S(n + 1) = \underbrace{1 + 3 + 5 + \cdots + (2n - 1)}_{S(n)} + (2n + 1) = S(n) + 2n + 1.$$

But by induction hypothesis,  $S(n) = n^2$ , hence:

$$S(n + 1) = n^2 + 2n + 1 = (n + 1)^2.$$

This completes the induction, and shows that the property is true for all positive integers.  $\square$