MATHEMATICAL INDUCTION

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Mathematical Induction (Summary)

This is a powerful method to prove properties of positive integers.

1. **Principle of Mathematical Induction.** Let P be a property of positive integers such that:

(1) *Base Case*: $P(1)$ is true, and

(2) *Inductive Step*: if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all positive integers.

The premise $P(n)$ in the inductive step is called *Induction Hypothesis*.

The validity of the Principle of Mathematical Induction is obvious. The base case states that $P(1)$ is true. Then the inductive step implies that $P(2)$ is also true. By the inductive step again we see that $P(3)$ is true, and so on. Consequently the property must be true for all positive integers:

$$\overset{\substack{\text{base case}\\ \downarrow}}{\underset{\text{true}}{P(1)}} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(2)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(3)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \cdots$$

$$\cdots \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(n)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(n+1)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \cdots$$

Example: Prove that the sum of the n first odd positive integers is n^2, i.e., $1+3+5+\cdots+(2n-1)=n^2$.

Answer: Let $S(n)=1+3+5+\cdots+(2n-1)$. We want to prove by induction that for every positive integer n, $S(n)=n^2$.

(1) *Base Case*: If $n = 1$ we have $S(1) = 1 = 1^2$, so the property is true for 1.

(2) *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for $n+1$, i.e., $S(n+1) = (n+1)^2$. In fact:

$$S(n+1) = \underbrace{1 + 3 + 5 + \cdots + (2n-1)}_{S(n)} + (2n+1) = S(n) + 2n + 1\,.$$

But by induction hypothesis, $S(n) = n^2$, hence:

$$S(n+1) = n^2 + 2n + 1 = (n+1)^2\,.$$

This completes the induction, and shows that the property is true for all positive integers. □