Putnam Test

All answers are real numbers, except for the last three questions, for which the answer is Y/N, question 5, for which the answer is an expression in terms of A, and question 10, for which the answer is a pair of integers. Give yourself two hours, and answer as many questions as you can. Marks will be deducted for incorrect answers, so don't guess. Use only pen and paper, no calculators, computers, etc. Don't Cheat! The questions are not necessarily in increasing difficulty. The last question is for survey purposes only.

1. The product N of three positive integers is 6 times their sum, and one of the integers is the sum of the other two. Find the sum of all possible values of N.

2. Find the eighth term of the sequence $1440, 1716, 1848, \ldots$, whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.

3. 7 short sighted people drop their glasses. In a mad scramble, each of them picks up a random pair of glasses. What is the probability that nobody picks up their own glasses?

4. Compute $\displaystyle\int_0^\infty \frac{\log(x)}{x^2+1}dx$

5. If $A = \displaystyle\int_0^\infty \frac{\sin x}{x}dx$, then what is $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx$ in terms of A?

6. Find the smallest integer that cannot be written as a square minus a prime.

7. Suppose that $\alpha+\beta+\gamma = 7$, $\alpha^2+\beta^2+\gamma^2 = 13$, and $\alpha^3+\beta^3+\gamma^3 = 19$. What is $\alpha\beta\gamma$?

8. Write $100 = a_1 + a_2 + \ldots + a_n$ as a sum of positive integers that maximizes the product $\prod a_i$. Find $\sum a_n^2$.

9. Evaluate $\displaystyle\sum_{n=1}^{2008} \frac{1}{\sqrt[3]{n^2} + \sqrt[3]{n(n+1)} + \sqrt[3]{(n+1)^2}}$.

10. Let $\phi = \dfrac{1+\sqrt{5}}{2}$. For a real number x, let $[x]$ denote the closest integer (above or below) to x. Let $f(n)$ be the last digit of $[\phi^n]$. What are $f(2008)$ and $f(2009)$?

11. Let $x_0 = 2$ and $y_0 = 3$. Suppose that $x_{n+1} = 2^{x_n}$ and $y_{n+1} = 3^{y_n}$, so, for example, x_0, x_1, x_2, and x_3 equal $2, 4, 16$ and 65536 respectively, while y_0, y_1, y_2, and y_0 equal $3, 27, 7625597484987$, and $3^{7625597484987}$. What is the smallest n such that $y_n > x_{100}$?

12. Can you make the Monday 5-6 problem session?

13. Can you make the Wednesday 4:30-5:30 problem session?

14. Do you have any problem solving competition experience?