


PUTNAM TRAINING MATHEMATICAL INDUCTION

(Last updated: September 19, 2022)

REMARK. This is a list of exercises on mathematical induction. —Miguel A. Lerma

EXERCISES

1. Prove that $n! > 2^n$ for all $n \geq 4$.
2. Prove that for any integer $n \geq 1$, $2^{2^n} - 1$ is divisible by 3.
3. Let a and b two distinct integers, and n any positive integer. Prove that $a^n - b^n$ is divisible by $a - b$.
4. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.
5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n , $r^n + 1/r^n$ is an integer.
6. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.
7. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.
8. A great circle is a circle drawn on a sphere that is an “equator”, i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?
9. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.
10. A chessboard is a 8×8 grid (64 squares arranged in 8 rows and 8 columns), but here we will call “chessboard” any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ ($n \geq 1$) defective chessboard can be tiled (completely covered without overlapping) with L-shaped trominos occupying exactly 3 squares, like this .

11. This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of chips). Two players arrange several piles of chips in a row. By turns each of them takes one chip from one of the piles and adds at will as many chips as he or she wishes to piles placed to the left of the pile from which the chip was taken. Assuming that the game ever finishes, the player that takes the last chip wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.
12. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.
13. Prove that for every $n \geq 2$, the expansion of $(1 + x + x^2)^n$ contains at least one even coefficient.
14. We define recursively the *Ulam numbers* by setting $u_1 = 1$, $u_2 = 2$, and for each subsequent integer n , we set n equal to the next Ulam number if it can be written uniquely as the sum of two different Ulam numbers; e.g.: $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, etc. Prove that there are infinitely many Ulam numbers.
15. Prove Bernoulli's inequality, which states that if $x > -1$, $x \neq 0$ and n is a positive integer greater than 1, then $(1 + x)^n > 1 + nx$.
16. Prove that two consecutive Fibonacci numbers are always relatively prime.
17. Let a_n be the following expression with n nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}$$

Prove that $a_n = 2 \cos \frac{\pi}{2^{n+1}}$.

HINTS

1. —
2. For the induction step, rewrite $2^{2(n+1)} - 1$ as a sum of two terms that are divisible by 3.
3. For the inductive step assume that step $a^n - b^n$ is divisible by $a - b$ and rewrite $a^{n+1} - b^{n+1}$ as a sum of two terms, one of them involving $a^n - b^n$ and the other one being a multiple of $a - b$.
4. Strong induction.
5. Rewrite $r^{n+1} + 1/r^{n+1}$ in terms of $r^k + 1/r^k$ with $k \leq n$.
6. How many regions can be intersected by the $(n + 1)$ th line?
7. Color a plane divided with n of lines in the desired way, and think how to recolor it after introducing the $(n + 1)$ th line.
8. How many regions can be intersected the by $(n + 1)$ th circle?
9. We have $1 = 5 \cdot (-7) + 12 \cdot 6 = 5 \cdot 5 + 12 \cdot (-2)$. Also, prove that if $n = 5x + 12y \geq 44$, then either $x \geq 7$ or $y \geq 2$.
10. For the inductive step, consider a $2^{n+1} \times 2^{n+1}$ defective chessboard and divide it into four $2^n \times 2^n$ chessboards. One of them is defective. Can the other three be made defective by placing strategically an L?
11. Use induction on the number of piles.
12. The numbers 8 and 9 form one such pair. Given a pair $(n, n + 1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.
13. Look at oddness/evenness of the four lowest degree terms of the expansion.
14. Assume that the first m Ulam numbers have already been found, and determine how the next Ulam number (if it exists) can be determined.
15. We have $(1 + x)^{n+1} = (1 + x)^n(1 + x)$.
16. In the induction step use the recursive definition of the Fibonacci numbers.
17. Find a recursive definition of a_n .

SOLUTIONS

1. We prove it by induction. The basis step corresponds to $n = 4$, and in this case certainly we have $4! > 2^4$ ($24 > 16$). Next, for the induction step, assume the inequality holds for some value of $n \geq 4$, i.e., we assume $n! > 2^n$, and look at what happens for $n + 1$:

$$(n + 1)! = n!(n + 1) > 2^n(n + 1) > 2^n \cdot 2 = 2^{n+1}.$$

↑
by induction hypothesis

Hence the inequality also holds for $n + 1$. Consequently it holds for every $n \geq 4$.

2. For the basis step, we have that for $n = 1$ indeed $2^{2 \cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3. Next, for the inductive step, assume that $n \geq 1$ and $2^{2^n} - 1$ is divisible by 3. We must prove that $2^{2^{(n+1)}} - 1$ is also divisible by 3. We have

$$2^{2^{(n+1)}} - 1 = 2^{2^{n+2}} - 1 = 4 \cdot 2^{2^n} - 1 = 3 \cdot 2^{2^n} + (2^{2^n} - 1).$$

In the last expression the last term is divisible by 3 by induction hypothesis, and the first term is also a multiple of 3, so the whole expression is divisible by 3 and we are done.

3. By induction. For $n = 1$ we have that $a^1 - b^1 = a - b$ is indeed divisible by $a - b$. Next, for the inductive step, assume that $a^n - b^n$ is divisible by $a - b$. We must prove that $a^{n+1} - b^{n+1}$ is also divisible by $a - b$. In fact:

$$a^{n+1} - b^{n+1} = (a - b)a^n + b(a^n - b^n).$$

On the right hand side the first term is a multiple of $a - b$, and the second term is divisible by $a - b$ by induction hypothesis, so the whole expression is divisible by $a - b$.

4. We prove it by strong induction. First we notice that the result is true for $n = 0$ ($F_0 = 0 < 1 = 2^0$), and $n = 1$ ($F_1 = 1 < 2 = 2^1$). Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every k such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} < 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1},$$

↑
by induction hypothesis

and we are done.

5. We prove it by induction. For $n = 1$ the expression is indeed an integer. For $n = 2$ we have that $r^2 + 1/r^2 = (r + 1/r)^2 - 2$ is also an integer. Next assume that $n > 2$ and that the expression is an integer for $n - 1$ and n . Then we have

$$\left(r^{n+1} + \frac{1}{r^{n+1}}\right) = \left(r^n + \frac{1}{r^n}\right) \left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right),$$

hence the expression is also an integer for $n + 1$.

6. By experimentation we easily find:

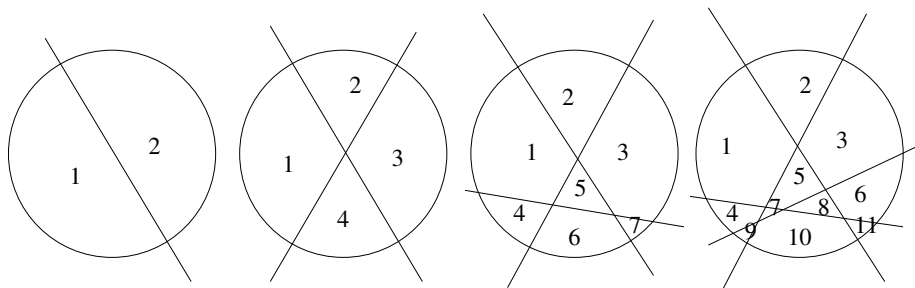


FIGURE 1. Plane regions.

n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n , i.e.:

$$R(n) = \frac{n^2 + n + 2}{2}.$$

We must prove that it is also true for $n + 1$, i.e.,

$$R(n + 1) = \frac{(n + 1)^2 + (n + 1) + 2}{2} = \frac{n^2 + 3n + 4}{2}.$$

So lets look at what happens when we introduce the $(n + 1)$ th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n + 1$ segments by those intersection points. Each of those $n + 1$ segments divides a previous region into two regions, so the number of regions increases by $n + 1$. Hence:

$$R(n + 1) = S(n) + n + 1.$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n + 1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2}.$$

QED.

7. We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other one in blue. Next assume that the regions obtained after dividing the plane with n lines can always be colored with two colors, red and blue, so that no two regions that share a boundary have the same color. We need to prove that such kind of coloring is also possible after dividing the plane with $n + 1$ lines. So assume that the plane divided by n lines has been colored in the desired way. After we introduce the $(n + 1)$ th line we need to recolor the plane to make sure that the new coloring still verifies that no

two regions that share a boundary have the same color. We do it in the following way. The $(n + 1)$ th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n + 1)$ th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

8. The answer is $f(n) = n^2 - n + 2$. The proof is by induction. For $n = 1$ we get $f(1) = 2$, which is indeed correct. Then we must prove that if $f(n) = n^2 - n + 2$ then $f(n + 1) = (n + 1)^2 - (n + 1) + 2$. In fact, the $(n + 1)$ th great circle meets each of the other great circles in two points each, so $2n$ points in total, which divide the circle into $2n$ arcs. Each of these arcs divides a region into two, so the number of regions grow by $2n$ after introducing the $(n + 1)$ th circle. Consequently $f(n + 1) = n^2 - n + 2 + 2n = n^2 + n + 2 = (n + 1)^2 - (n + 1) + 2$, QED.

9. We proceed by induction. For the basis step, i.e. $n = 44$, we can use four 5ϕ stamps and two 12ϕ stamps, so that $5 \cdot 4 + 12 \cdot 2 = 44$. Next, for the induction step, assume that for a given $n \geq 44$ the task is possible by using x 5ϕ stamps and y 12ϕ stamps, i.e. $n = 5x + 12y$. We must now prove that we can find some combination of x' 5ϕ stamps and y' 12ϕ stamps so that $n + 1 = 5x' + 12y'$. First note that either $x \geq 7$ or $y \geq 2$ — otherwise we would have $x \leq 6$ and $y \leq 1$, hence $n \leq 5 \cdot 6 + 12 \cdot 1 = 42 < 44$, contradicting the hypothesis that $n \geq 44$. So we consider the two cases:

1. If $x \geq 7$, then we can accomplish the goal by setting $x' = x - 7$ and $y' = y + 6$:

$$5x' + 12y' = 5(x - 7) + 12(y + 6) = 5x + 12y + 1 = n + 1.$$

2. On the other hand, if $y \geq 2$ then, we can do it by setting $x' = x + 5$ and $y' = y - 2$:

$$5x' + 12y' = 5(x + 5) + 12(y - 2) = 5x + 12y + 1 = n + 1.$$

10. We prove it by induction on n . For $n = 1$ the defective chessboard consists of just a single L and the tiling is trivial. Next, for the inductive step, assume that a $2^n \times 2^n$ defective chessboard can be tiled with L's. Now, given a $2^{n+1} \times 2^{n+1}$ defective chessboard, we can divide it into four $2^n \times 2^n$ chessboards as shown in the figure. One of them will have a square missing and will be defective, so it can be tiled with L's. Then we place an L covering exactly one corner of each of the other $2^n \times 2^n$ chessboards (see figure). The remaining part of each of those chessboards is like a defective chessboard and can be tiled in the desired way too. So the whole $2^{n+1} \times 2^{n+1}$ defective chessboard can be tiled with L's.

11. We use induction on the number n of piles. For $n = 1$ we have only one pile, and since each player must take at least one chip from that pile, the number of tokens in it will decrease at each move until it is empty. Next, for the induction step, assume that the game with n piles must end eventually. We will prove that the same is true for $n + 1$ piles. First note that the players cannot keep taking chips only from the

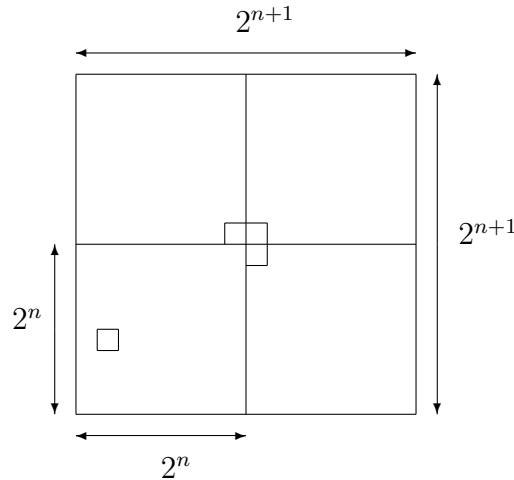


FIGURE 2. A $2^{n+1} \times 2^{n+1}$ defective chessboard.

first n piles, since by induction hypothesis the game with n piles eventually ends. So sooner or later one player must take a chip from the $(n + 1)$ th pile. It does not matter how many tokens he or she adds to the other n piles after that, it is still true that the players cannot keep taking chips only from the first n piles forever, so eventually someone will take another chip from the $(n + 1)$ th pile. Consequently, the number of chips in that pile will continue decreasing until it is empty. After that we will have only n piles left, and by induction hypothesis the game will end in finitely many steps after that.¹

- 12. The numbers 8 and 9 are a pair of consecutive square-fulls. Next, if n and $n + 1$ are square-full, so are $4n(n + 1)$ and $4n(n + 1) + 1 = (2n + 1)^2$.
- 13. For $n = 2, 3, 4, 5, 6$ we have:

$$\begin{aligned} (1 + x + x^2)^2 &= 1 + 2x + 3x^2 + 2x^3 + x^4 \\ (1 + x + x^2)^3 &= 1 + 3x + 6x^2 + 7x^3 + \dots \\ (1 + x + x^2)^4 &= 1 + 4x + 10x^2 + 16x^3 + \dots \\ (1 + x + x^2)^5 &= 1 + 5x + 15x^2 + 30x^3 + \dots \\ (1 + x + x^2)^6 &= 1 + 6x + 21x^2 + 50x^3 + \dots \end{aligned}$$

In general, if $(1 + x + x^2)^n = a + bx + cx^2 + dx^3 + \dots$, then

$$(1 + x + x^2)^{n+1} = a + (a + b)x + (a + b + c)x^2 + (b + c + d)x^3 + \dots,$$

¹An alternate proof based on properties of ordinal numbers is as follows (requires some advanced set-theoretical knowledge.) Here ω = first infinite ordinal number, i.e., the first ordinal after the sequence of natural numbers $0, 1, 2, 3, \dots$. Let the ordinal number $\alpha = a_0 + a_1\omega + a_2\omega^2 + \dots + a_{n-1}\omega^{n-1}$ represent a configuration of n piles with a_0, a_1, \dots, a_{n-1} chips respectively (read from left to right). After a move the ordinal number representing the configuration of chips always decreases. Every decreasing sequence of ordinal numbers is finite. Hence the result.

hence the first four coefficients of $(1 + x + x^2)^{n+1}$ depend only on the first four coefficients of $(1 + x + x^2)^n$. The same is true if we write the coefficients modulo 2, i.e., as “0” if they are even, or “1” if they are odd. So, if we call $q_n(x) = (1 + x + x^2)^n$ with the coefficients written modulo 2, we have

$$\begin{aligned} q_1(x) &= 1 + 1x + 1x^2 \\ q_2(x) &= 1 + 0x + 1x^2 + 0x^3 + 1x^4 \\ q_3(x) &= 1 + 1x + 0x^2 + 1x^3 + \cdots \\ q_4(x) &= 1 + 0x + 0x^2 + 0x^3 + \cdots \\ q_5(x) &= 1 + 1x + 1x^2 + 0x^3 + \cdots \\ q_6(x) &= 1 + 0x + 1x^2 + 0x^3 + \cdots \end{aligned}$$

We notice that the first four coefficients of $q_6(x)$ coincide with those of $q_2(x)$, and since these first four coefficients determine the first four coefficients of each subsequent polynomial of the sequence, they will repeat periodically so that those of $q_n(x)$ will always coincide with those of q_{n+4} . Since for $n = 2, 3, 4, 5$ at least one of the first four coefficients of $q_n(x)$ is 0 (equivalently, at least one of the first four coefficients of $(1 + x + x^2)^n$ is even), the same will hold for all subsequent values of n .

14. Let $U_m = \{u_1, u_2, \dots, u_m\}$ ($m \geq 2$) be the first m Ulam numbers (written in increasing order). Let S_m be the set of integers greater than u_m that can be written uniquely as the sum of two different Ulam numbers from U_m . The next Ulam number u_{m+1} is precisely the minimum element of S_m , unless S_m is empty, but it is not because $u_{m-1} + u_m \in S_m$.
15. By induction. For the base case $n = 2$ the inequality is $(1 + x)^2 > 1 + 2x$, obviously true because $(1 + x)^2 - (1 + 2x) = x^2 > 0$. For the induction step, assume that the inequality is true for n , i.e., $(1 + x)^n > 1 + nx$. Then, for $n + 1$ we have

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n(1 + x) > (1 + nx)(1 + x) = \\ &1 + (n + 1)x + x^2 > 1 + (n + 1)x, \end{aligned}$$

and the inequality is also true for $n + 1$.

16. This can be proved easily by induction. Base case: $F_1 = 1$ and $F_2 = 1$ are in fact relatively prime. Induction Step: we must prove that if F_n and F_{n+1} are relatively prime then so are F_{n+1} and F_{n+2} . But this follows from the recursive definition of the Fibonacci sequence: $F_n + F_{n+1} = F_{n+2}$; any common factor of F_{n+1} and F_{n+2} would be also a factor of F_n , and consequently it would be a common factor of F_n and F_{n+1} (which by induction hypothesis are relatively prime.)
17. Note that a_n can be defined recursively like this: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. We proceed by induction. For $n = 1$ we have in fact $a_1 = \sqrt{2}$, and $2 \cos \frac{\pi}{4} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$.

Next, assuming the result is true for some $n \geq 1$, we have

$$\begin{aligned} a_{n+1} &= \sqrt{2 + a_n} = \sqrt{2 + 2 \cos \frac{\pi}{2^{n+1}}} \\ &= \sqrt{2 + 2 \cos 2 \frac{\pi}{2^{n+2}}} \\ &= \sqrt{2 + 2(2 \cos^2 \frac{\pi}{2^{n+2}} - 1)} \\ &= \sqrt{4 \cos^2 \frac{\pi}{2^{n+2}}} \\ &= 2 \cos \frac{\pi}{2^{n+2}}, \end{aligned}$$