

PUTNAM TRAINING INEQUALITIES

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REMARK. This is a list of exercises on inequalities. —Miguel A. Lerma

EXERCISES

1. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.
2. Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \dots$,
3. If $0 < p, 0 < q$, and $p + q < 1$, prove that $(px + qy)^2 \leq px^2 + qy^2$.
4. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.
5. Let $x, y, z > 0$ with $xyz = 1$. Prove that $x + y + z \leq x^2 + y^2 + z^2$.
6. Show that

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \dots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + a_2 + \dots + a_n)^2 + (b_1 + b_2 + \dots + b_n)^2}$$

7. Find the minimum value of the function $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, where x_1, x_2, \dots, x_n are positive real numbers such that $x_1x_2 \cdots x_n = 1$.
8. Let $x, y, z \geq 0$ with $xyz = 1$. Find the minimum of

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

9. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

10. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}.$$

11. (Putnam, 2003) Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}$$

12. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$. What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?
13. Which is larger, 1999^{1999} or 2000^{1998} ?
14. Which is larger, $\log_2 3$ or $\log_3 5$?
15. Prove that there are no positive integers a, b such that $b^2 + b + 1 = a^2$.
16. (Inspired in Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.
17. (Putnam 1984) Find the minimum value of

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v} \right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

18. Show that $\frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$.

19. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

20. Let a_1, a_2, \dots, a_n be a sequence of positive numbers, and let b_1, b_2, \dots, b_n be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

21. (Rearrangement Inequality.) Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n increasing sequences of real numbers, and let x_1, x_2, \dots, x_n be any permutation of b_1, b_2, \dots, b_n . Show that

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n a_i x_i.$$

22. Prove that the p -mean tends to the geometric mean as p approaches zero. In other words, if a_1, \dots, a_n are positive real numbers, then

$$\lim_{p \rightarrow 0} \left(\frac{1}{n} \sum_{k=1}^n a_k^p \right)^{1/p} = \left(\prod_{k=1}^n a_k \right)^{1/n}$$

23. If a , b , and c are the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3.$$

24. Here we use Knuth's up-arrow notation: $a \uparrow b = a^b$, $a \uparrow\uparrow b = \underbrace{a \uparrow (a \uparrow (\dots \uparrow a))}_{b \text{ copies of } a}$, so

e.g. $2 \uparrow\uparrow 3 = 2 \uparrow (2 \uparrow 2) = 2^{2^2}$. What is larger, $2 \uparrow\uparrow 2011$ or $3 \uparrow\uparrow 2010$?

25. Prove that $e^{1/e} + e^{1/\pi} \geq 2e^{1/3}$.

26. Prove that the function $f(x) = \sum_{i=1}^n (x - a_i)^2$ attains its minimum value at $x = \bar{a} = \frac{a_1 + \dots + a_n}{n}$.

27. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4, \quad x_2 + \frac{1}{x_3} = 1, \dots, \quad x_{99} + \frac{1}{x_{100}} = 4, \quad x_{100} + \frac{1}{x_1} = 1.$$

28. Prove that if the numbers a , b , and c satisfy the inequalities $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then one of those numbers is the sum of the other two.

29. Find the minimum of $\sin^3 x / \cos x + \cos^3 x / \sin x$, $0 < x < \pi/2$.

30. Let $a_i > 0$, $i = 1, \dots, n$, and $s = a_1 + \dots + a_n$. Prove

$$\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \dots + \frac{a_n}{s - a_n} \geq \frac{n}{n - 1}.$$

31. Find the maximum value of $f(x) = \sin^4(x) + \cos^4 x$ for $x \in \mathbb{R}$

32. Let a, b, c be positive real numbers. Prove that $2 \left(\frac{a+b}{2} - \sqrt{ab} \right) \leq 3 \left(\frac{a+b+c}{3} - \sqrt[3]{abc} \right)$. When is equality attained?

33. (Generalization of previous problem.) Let a_n , $n = 1, 2, 3, \dots$ be a sequence of positive real numbers. Prove that $b_n = a_1 + \dots + a_n - n \sqrt[n]{a_1 \cdots a_n}$ is an increasing sequence of non-negative real numbers.

34. Let $f : [0, \infty) \rightarrow \mathbb{R}$ a function with the following properties:

(a) f is subadditive, i.e., $f(x + y) \leq f(x) + f(y)$, for every $x, y \in [0, \infty)$.

(b) f is (right) differentiable at 0.

(c) $f(0) = 0$.

Prove $f(x) \leq x f'(0)$ for every $x \in [0, \infty)$.

35. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$, with $a_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n-1\}$, be a polynomial with roots $x_1, x_2, \dots, x_n \in \mathbb{R}$, and let $x_{i_0} = \max_{1 \leq i \leq n} x_i$. Prove that if $x \geq x_{i_0}$, then $P'(x) \geq n \sqrt[n]{(P(x))^{n-1}}$.

36. Prove that if a, b, c, d are positive real numbers, then $\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+d)(b+c)}$.

37. Let N be a positive integer and let S_N be the sum

$$S_N = \frac{1}{2} \sum_{k=1}^{N^2} \frac{1}{\sqrt{k}}.$$

Prove that $N - 1 < S_N < N$.

38. Prove $\int_0^{\frac{\pi}{2}} e^{\sin x} dx \geq \frac{\pi}{2}(e - 1)$.

HINTS

1. One way to solve this problem is by using the Arithmetic Mean-Geometric Mean inequality on each factor of the left hand side.
2. Apply the Arithmetic Mean-Geometric Mean inequality to the set of numbers $1, 2, \dots, n$.
3. Power means inequality with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$.
4. Power means inequality.
5. —
6. This problem can be solved by using Minkowski's inequality, but another way to look at it is by an appropriate geometrical interpretation of the terms (as distances between points of the plane.)
7. Many minimization or maximization problems are inequalities in disguise. The solution usually consists of "guessing" the maximum or minimum value of the function, and then proving that it is in fact maximum or minimum. In this case, given the symmetry of the function a good guess is $f(1, 1, \dots, 1) = n$, so try to prove $f(x_1, x_2, \dots, x_n) \geq n$. Use the Arithmetic Mean-Geometric Mean inequality on x_1, \dots, x_n .
8. Apply the Cauchy-Schwarz inequality to the vectors $(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{z+x}}, \frac{z}{\sqrt{x+y}})$ and (u, v, w) , and choose appropriate values for u, v, w .
9. Arithmetic-Harmonic Mean inequality.
10. Assume $a \leq b \leq c$, $A \leq B \leq C$, and use Chebyshev's Inequality.
11. Divide by the right hand side and use the Arithmetic Mean-Geometric Mean inequality on both terms of the left.
12. Note that $n!$ is increasing ($n < m \implies n! < m!$)
13. Look at the function $f(x) = (1999 - x) \ln(1999 + x)$.
14. Use the definition of logarithm.
15. The numbers b^2 and $(b + 1)^2$ are consecutive squares.
16. Use the Arithmetic Mean-Geometric Mean inequality on the squares of the roots of the polynomial.
17. Think geometrically. Interpret the given expression as the square of the distance between two points in the plane. The problem becomes that of finding the minimum distance between two curves.

18. Consider the expressions $P = \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right)$ and $Q = \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \cdots \left(\frac{2n-2}{2n-1}\right)$. Note that $\frac{k-1}{k} < \frac{k}{k+1}$, for $k = 1, 2, \dots$.
19. Look at the binomial expansion of $(m+n)^{m+n}$.
20. Arithmetic Mean-Geometric Mean inequality.
21. Try first the cases $n = 1$ and $n = 2$. Then use induction.
22. Take logarithms and use L'Hôpital.
23. Set $x = b + c - a$, $y = c + a - b$, $z = a + b - c$.
24. We have $2^{2^2} = 16 < 27 = 3^3$.
25. Show that $f(x) = e^{1/x}$ for $x > 0$ is decreasing and convex.
26. Prove that $f(x) - f(\bar{a}) \geq 0$.
27. By the AM-GM inequality we have $x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \dots$. Try to prove that those inequalities are actually equalities.
28. Square both sides of those inequalities.
29. Rearrangement inequality.
30. Rearrangement inequality.
31. Note that $|\sin x| \leq 1$, so what which is smaller, $\sin^2 x$ or $\sin^4 x$? (Same with $\cos x$.)
32. Arithmetic Mean-Geometric Mean inequality.
33. Arithmetic Mean-Geometric Mean inequality.
34. Look at the limit of $\frac{f(x/n)}{x/n}$ as $n \rightarrow \infty$.
35. Factor the polynomial, differentiate it, and use the AM-GM inequality.
36. Trigonometric substitution.
37. Telescoping.
38. Concavity.

SOLUTIONS

1. Using the Arithmetic Mean-Geometric Mean Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3}\right) \left(\frac{ab^2 + bc^2 + ca^2}{3}\right) \geq \left(\sqrt[3]{a^3b^3c^3}\right) \left(\sqrt[3]{a^3b^3c^3}\right) = a^2b^2c^2.$$

Multiplying by 9 we get the desired inequality.

Another solution consists of using the Cauchy-Schwarz inequality:

$$\begin{aligned} (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) &= \\ &= \left((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2\right) \left((\sqrt{b}c)^2 + (\sqrt{c}a)^2 + (\sqrt{a}b)^2\right) \\ &\geq (abc + abc + abc)^2 \\ &= 9a^2b^2c^2. \end{aligned}$$

2. This result is the Arithmetic Mean-Geometric Mean applied to the set of numbers $1, 2, \dots, n$:

$$\sqrt[n]{1 \cdot 2 \cdot \dots \cdot n} < \frac{1 + 2 + \dots + n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}.$$

Raising both sides to the n th power we get the desired result.

3. The simplest solution consists of using the weighted power means inequality to the (weighted) arithmetic and quadratic means of x and y with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$:

$$\frac{p}{p+q}x + \frac{q}{p+q}y \leq \sqrt{\frac{p}{p+q}x^2 + \frac{q}{p+q}y^2},$$

hence

$$(px + qy)^2 \leq (p+q)(px^2 + qy^2).$$

Or we can use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} (px + qy)^2 &= (\sqrt{p}\sqrt{p}x + \sqrt{q}\sqrt{q}y)^2 \\ &\leq (\{\sqrt{p}\}^2 + \{\sqrt{q}\}^2) (\{\sqrt{p}x\}^2 + \{\sqrt{q}y\}^2) \quad (\text{Cauchy-Schwarz}) \\ &= (p+q)(px^2 + qy^2). \end{aligned}$$

Finally we use $p+q \leq 1$ to obtain the desired result.

4. By the power means inequality:

$$\underbrace{\frac{a+b+c}{3}}_{M^1(a,b,c)} \geq \underbrace{\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2}_{M^{1/2}(a,b,c)}$$

From here the desired result follows.

5. We have:

$$\begin{aligned} x + y + z &= (x + y + z)\sqrt[3]{xyz} && (xyz = 1) \\ &\leq \frac{(x + y + z)^2}{3} && \text{(AM-GM inequality)} \\ &\leq x^2 + y^2 + z^2. && \text{(power means inequality)} \end{aligned}$$

6. The result can be obtained by using Minkowski's inequality repeatedly:

$$\begin{aligned} \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2} &\geq \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} + \cdots + \sqrt{a_n^2 + b_n^2} \\ &\geq \sqrt{(a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2} + \cdots \\ &\quad + \sqrt{a_n^2 + b_n^2} \\ &\quad \dots \\ &\geq \sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2} \end{aligned}$$

Another way to think about it is geometrically. Consider a sequence of points in the plane $P_k = (x_k, y_k)$, $k = 0, \dots, n$, such that

$$(x_k, y_k) = (x_{k-1} + a_k, y_{k-1} + b_k) \quad \text{for } k = 1, \dots, n.$$

Then the left hand side of the inequality is the sum of the distances between two consecutive points, while the right hand side is the distance between the first one and the last one:

$$d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{n-1}, P_n) \leq d(P_0, P_n).$$

7. By the Arithmetic Mean-Geometric Mean Inequality

$$1 = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

Hence $f(x_1, x_2, \dots, x_n) \geq n$. On the other hand $f(1, 1, \dots, 1) = n$, so the minimum value is n .

8. For $x = y = z = 1$ we see that $S = 3/2$. We will prove that in fact $3/2$ is the minimum value of S by showing that $S \geq 3/2$.

Note that

$$S = \left(\frac{x}{\sqrt{y+z}} \right)^2 + \left(\frac{y}{\sqrt{z+x}} \right)^2 + \left(\frac{z}{\sqrt{x+y}} \right)^2.$$

Hence by the Cauchy-Schwarz inequality:

$$S \cdot (u^2 + v^2 + w^2) \geq \left(\frac{xu}{\sqrt{y+z}} + \frac{yv}{\sqrt{z+x}} + \frac{zw}{\sqrt{x+y}} \right)^2.$$

Writing $u = \sqrt{y+z}$, $v = \sqrt{z+x}$, $w = \sqrt{x+y}$ we get

$$S \cdot 2(x + y + z) \geq (x + y + z)^2,$$

hence, dividing by $2(x + y + z)$ and using the Arithmetic Mean-Geometric Mean inequality:

$$S \geq \frac{1}{2}(x + y + z) \geq \frac{1}{2} \cdot 3\sqrt[3]{xyz} = \frac{3}{2}.$$

9. By the Arithmetic Mean-Harmonic Mean inequality:

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x + y + z}{3} = \frac{1}{3},$$

hence

$$9 \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

On the other hand for $x = y = z = 1/3$ the sum is 9, so the minimum value is 9.

10. Assume $a \leq b \leq c$, $A \leq B \leq C$. Then

$$\begin{aligned} 0 &\leq (a - b)(A - B) + (a - c)(A - C) + (b - c)(B - C) \\ &= 3(aA + bB + cC) - (a + b + c)(A + B + C). \end{aligned}$$

Using $A + B + C = \pi$ and dividing by $3(a + b + c)$ we get the desired result.

- *Remark:* We could have used also Chebyshev's Inequality:

$$\frac{aA + bB + cC}{3} \geq \left(\frac{a + b + c}{3}\right) \left(\frac{A + B + C}{3}\right).$$

11. Assume $a_i + b_i > 0$ for each i (otherwise both sides are zero). Then by the Arithmetic Mean-Geometric Mean inequality

$$\left(\frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)}\right)^{1/n} \leq \frac{1}{n} \left(\frac{a_1}{a_1 + b_1} + \cdots + \frac{a_n}{a_n + b_n}\right),$$

and similarly with the roles of a and b reversed. Adding both inequalities and clearing denominators we get the desired result.

(*Remark:* The result is known as *superadditivity* of the *geometric mean*.)

12. We have that $n!$ is increasing for $n \geq 1$, i.e., $1 \leq n < m \implies n! < m!$ So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

13. Consider the function $f(x) = (1999 - x) \ln(1999 + x)$. Its derivative is $f'(x) = -\ln(1999 + x) + \frac{1999 - x}{1999 + x}$, which is negative for $0 \leq x \leq 1$, because in that interval

$$\frac{1999 - x}{1999 + x} \leq 1 = \ln e < \ln(1999 + x).$$

Hence f is decreasing in $[0, 1]$ and $f(0) > f(1)$, i.e., $1999 \ln 1999 > 1998 \ln 2000$. Consequently $1999^{1999} > 2000^{1998}$.

14. Let $x = \log_2 3$ and $y = \log_3 5$, so $2^x = 3$, $3^y = 5$. Then, $27 = 3^3 = (2^x)^3 = 8^x$, and $25 = 5^2 = (3^y)^2 = 9^y$, hence $8^x > 9^y$, but $8 < 9$, hence $x > y$, i.e., $\log_2 3 > \log_3 5$.

15. We have $b^2 < \underbrace{b^2 + b + 1}_{a^2} < b^2 + 2b + 1 = (b + 1)^2$. But b^2 and $(b + 1)^2$ are consecutive squares, so there cannot be a square strictly between them.
16. We may assume that the leading coefficient is $+1$. The sum of the squares of the roots of $x^n + a_1x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is a_n^2 . Using the Arithmetic Mean-Geometric Mean inequality we have

$$\frac{a_1^2 - 2a_2}{n} \geq \sqrt[n]{a_n^2}.$$

Since the coefficients are ± 1 that inequality is $(1 \pm 2)/n \geq 1$, hence $n \leq 3$.

Remark: $x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$ is an example of 3th degree polynomial with all coefficients equal to ± 1 and only real roots.

17. The given function is the square of the distance between a point of the quarter of circle $x^2 + y^2 = 2$ in the open first quadrant and a point of the half hyperbola $xy = 9$ in that quadrant. The tangents to the curves at $(1, 1)$ and $(3, 3)$ separate the curves, and both are perpendicular to $x = y$, so those points are at the minimum distance, and the answer is $(3 - 1)^2 + (3 - 1)^2 = 8$.

18. Let

$$P = \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right), \quad Q = \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \cdots \left(\frac{2n-2}{2n-1}\right).$$

We have $PQ = \frac{1}{2n}$. Also $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \cdots < \frac{2n-1}{2n}$, hence $2P \geq Q$, so $2P^2 \geq PQ = \frac{1}{2n}$, and from here we get $P \geq \frac{1}{\sqrt{4n}}$.

On the other hand we have $P < Q \frac{2n}{2n+1} < Q$, hence $P^2 < PQ = \frac{1}{2n}$, and from here $P < \frac{1}{\sqrt{2n}}$.

19. The given inequality is equivalent to

$$\frac{(m+n)!}{m!n!} m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n},$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

20. Using the Arithmetic Mean-Geometric Mean inequality we get:

$$\frac{1}{n} \left\{ \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \right\} \geq \sqrt[n]{\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdots \frac{a_n}{b_n}} = 1.$$

From here the desired result follows.

- 21.** We prove it by induction. For $n = 1$ the result is trivial, and for $n = 2$ it is a simple consequence of the following:

$$0 \leq (a_2 - a_1)(b_2 - b_1) = (a_1b_1 + a_2b_2) - (a_1b_2 + a_2b_1).$$

Next assume that the result is true for some $n \geq 2$. We will prove that it is true for $n + 1$. There are two possibilities:

1. If $x_{n+1} = b_{n+1}$, then we can apply the induction hypothesis to the n first terms of the sum and we are done.
2. If $x_{n+1} \neq b_{n+1}$, then $x_j = b_{n+1}$ for some $j \neq n + 1$, and $x_{n+1} = b_k$ for some $k \neq n + 1$. Hence:

$$\begin{aligned} \sum_{i=1}^{n+1} a_i x_i &= \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j x_j + a_{n+1} x_{n+1} \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j b_{n+1} + a_{n+1} b_k \end{aligned}$$

(using the inequality for the two-term increasing sequences a_j, a_{n+1} and b_k, b_{n+1})

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j b_k + a_{n+1} b_{n+1}.$$

This reduces the problem to case 1.

- 22.** We have

$$\ln \left(\frac{1}{n} \sum_{k=1}^n a_k^p \right)^{1/p} = \frac{\ln \left(\frac{1}{n} \sum_{k=1}^n a_k^p \right)}{p}.$$

Also, $a_k \rightarrow 1$ as $p \rightarrow 0$, hence numerator and denominator tend to zero as p approaches zero. Using L'Hôpital we get

$$\lim_{p \rightarrow 0} \frac{\ln \left(\frac{1}{n} \sum_{k=1}^n a_k^p \right)}{p} = \lim_{p \rightarrow 0} \frac{\sum_{k=1}^n a_k^p \ln a_k}{\sum_{k=1}^n a_k^p} = \frac{\sum_{k=1}^n \ln a_k}{n} = \ln \left(\prod_{k=1}^n a_k \right)^{1/n}.$$

From here the desired result follows.

- 23.** Set $x = b + c - a$, $y = c + a - b$, $z = a + b - c$. The triangle inequality implies that x , y , and z are positive. Furthermore, $a = (y + z)/2$, $b = (z + x)/2$, and $c = (x + y)/2$. The LHS of the inequality becomes:

$$\frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} = \frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{x}{z} + \frac{z}{x} \right) \geq 3.$$

- 24.** We have that $2 \uparrow\uparrow 3 = 2^{2^2} = 16 < 27 = 3^3 = 3 \uparrow\uparrow 2$. Then using $a \uparrow\uparrow (n+1) = a^{a \uparrow\uparrow n}$ we get $2 \uparrow\uparrow (n+1) < 3 \uparrow\uparrow n$ for $n \geq 2$, and from here it follows that $2 \uparrow\uparrow 2011 < 3 \uparrow\uparrow 2010$.
- 25.** Consider the function $f(x) = e^{1/x}$ for $x > 0$. We have $f'(x) = -\frac{1}{x^2}e^{1/x} < 0$, $f''(x) = e^{1/x}(\frac{2}{x^3} + \frac{1}{x^4}) > 0$, hence f is decreasing and convex. By convexity, we have

$$\frac{1}{2}(f(e) + f(\pi)) \geq f\left(\frac{e+\pi}{2}\right).$$

On the other hand we have $(e + \pi)/2 < 3$, and since f is decreasing, $f(\frac{e+\pi}{2}) > f(3)$, and from here the result follows.

- 26.** We have:

$$\begin{aligned} f(x) - f(\bar{a}) &= \sum_{i=1}^n (x - a_i)^2 - \sum_{i=1}^n (\bar{a} - a_i)^2 \\ &= \sum_{i=1}^n \{(x - a_i)^2 - (\bar{a} - a_i)^2\} \\ &= \sum_{i=1}^n (x^2 - 2a_i x - \bar{a}^2 + 2a_i \bar{a}) \\ &= nx^2 - 2n\bar{a}x + n\bar{a}^2 \\ &= n(x - \bar{a})^2 \geq 0, \end{aligned}$$

hence $f(x) \geq f(\bar{a})$ for every x .

- 27.** By the Geometric Mean-Arithmetic Mean inequality

$$x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \dots, x_{100} + \frac{1}{x_1} \geq 2\sqrt{\frac{x_{100}}{x_1}}.$$

Multiplying we get

$$\left(x_1 + \frac{1}{x_2}\right) \left(x_2 + \frac{1}{x_3}\right) \cdots \left(x_{100} + \frac{1}{x_1}\right) \geq 2^{100}.$$

From the system of equations we get

$$\left(x_1 + \frac{1}{x_2}\right) \left(x_2 + \frac{1}{x_3}\right) \cdots \left(x_{100} + \frac{1}{x_1}\right) = 2^{100},$$

so all those inequalities are equalities, i.e.,

$$x_1 + \frac{1}{x_2} = 2\sqrt{\frac{x_1}{x_2}} \implies \left(\sqrt{x_1} - \frac{1}{\sqrt{x_2}}\right)^2 = 0 \implies x_1 = \frac{1}{x_2},$$

and analogously: $x_2 = 1/x_3, \dots, x_{100} = 1/x_1$. Hence $x_1 = 1/x_2, x_2 = 1/x_3, \dots, x_{100} = 1/x_1$, and from here we get $x_1 = 2, x_2 = 1/2, \dots, x_{99} = 2, x_{100} = 1/2$.

28. Squaring the inequalities and moving their left hand sides to the right we get

$$\begin{aligned} 0 &\geq c^2 - (a - b)^2 = (c + a - b)(c - a + b) \\ 0 &\geq a^2 - (b - c)^2 = (a + b - c)(a - b + c) \\ 0 &\geq b^2 - (c - a)^2 = (b + c - a)(b - c + a). \end{aligned}$$

Multiplying them together we get:

$$0 \geq (a + b - c)^2(a - b + c)^2(-a + b + c)^2,$$

hence, one of the factors must be zero.

29. The answer is 1. In fact, the sequences $(\sin^3 x, \cos^3 x)$ and $(1/\sin x, 1/\cos x)$ are oppositely sorted, hence by the rearrangement inequality:

$$\begin{aligned} \sin^3 x / \cos x + \cos^3 x / \sin x &\geq \sin^3 x / \sin x + \cos^3 x / \cos x \\ &= \sin^2 x + \cos^2 x = 1. \end{aligned}$$

Equality is attained at $x = \pi/4$.

30. By the rearrangement inequality we have for $k = 2, 3, \dots, n$ (assume $a_{n+i} = a_i$):

$$\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \dots + \frac{a_n}{s - a_n} \geq \frac{a_1}{s - a_k} + \frac{a_2}{s - a_{k+1}} + \dots + \frac{a_n}{s - a_{k-1}}.$$

Adding those $n - 1$ inequalities we get

$$(n - 1) \left(\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \dots + \frac{a_n}{s - a_n} \right) \geq \frac{s - a_1}{s - a_1} + \frac{s - a_2}{s - a_2} + \dots + \frac{s - a_n}{s - a_n} = n,$$

and the result follows.

31. The answer is 1. Since $|\sin x| \leq 1$ we have $\sin^4(x) \leq \sin^2(x)$, and analogously $\cos^4(x) \leq \cos^2(x)$. Hence $f(x) = \sin^4(x) + \cos^4(x) \leq \sin^2(x) + \cos^2(x) = 1$. On the other hand the value 1 is attained e.g. at $x = 0$.

32. Let $u = v = \sqrt{ab}$, $w = c$. By the AGM inequality we have

$$\sqrt[3]{uvw} \leq \frac{u + v + w}{3} \Rightarrow \sqrt[3]{abc} \leq \frac{2\sqrt{ab} + c}{3} \Rightarrow 3\sqrt[3]{abc} - (a + b + c) \leq 2\sqrt{ab} - (a + b).$$

The last inequality is equivalent to the desired result.

Equality happens precisely for $u = v = w$, i.e., $c = \sqrt{ab}$.

33. The AGM inequality applied to a_1, \dots, a_n shows that $b_n \geq 0$. Also, letting $u_k = \sqrt[n]{a_1 \cdots a_n}$, $k = 1, \dots, n$, $u_{n+1} = a_{n+1}$ and using again the AGM inequality we get

$$\begin{aligned} \sqrt[n+1]{u_1 \cdots u_n u_{n+1}} &\leq \frac{u_1 + \dots + u_n + u_{n+1}}{n + 1} \\ \sqrt[n+1]{a_1 \cdots a_n a_{n+1}} &\leq \frac{n \sqrt[n]{a_1 \cdots a_n} + a_{n+1}}{n + 1}. \end{aligned}$$

Multiplying both sides by $n + 1$ and subtracting $a_1 + \dots + a_n + a_{n+1}$ we get $-b_{n+1} \leq -b_n$, which is equivalent to the desired result.

34. By subadditivity we have that for every positive integer n and any $y > 0$, $f(ny) \leq nf(y)$. Writing $x = ny$, we get $f(x) \leq nf(x/n) = x \frac{f(x/n)}{x/n} \xrightarrow{n \rightarrow \infty} xf'(0)$, and the result follows.

35. First, factor the polynomial:

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Note that the hypothesis $x \geq x_{i_0}$ implies that all the factors are positive. Next, differentiate:

$$P'(x) = P_1(x) + P_2(x) + \cdots + P_n(x),$$

where $P_i(x) = P(x)/(x - x_i)$, $1 \leq i \leq n$. By the AM-GM inequality we have

$$\frac{1}{n} P'(x) = \frac{1}{n} \sum_{i=1}^n P_i(x) \geq \left(\prod_{i=1}^n P_i(x) \right)^{1/n}.$$

Also we have

$$\prod_{i=1}^n P_i = \frac{(P(x))^n}{P(x)} = (P(x))^{n-1},$$

and the result follows.

36. The inequality is equivalent to:

$$\sqrt{\frac{a}{a+d} \cdot \frac{b}{b+c}} + \sqrt{\frac{c}{b+c} \cdot \frac{d}{a+d}} \leq 1.$$

Setting $\frac{a}{a+d} = \sin^2 \alpha$, $\frac{b}{b+c} = \sin^2 \beta$ ($0 < \alpha, \beta < \frac{\pi}{2}$), the inequality becomes $\sin \alpha \sin \beta + \cos \alpha \cos \beta \leq 1$, i.e., $\cos(\alpha - \beta) \leq 1$.

37. For every $n \geq 1$ we have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n} + \sqrt{n+1}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n-1} + \sqrt{n}} = \sqrt{n} - \sqrt{n-1},$$

hence

$$\sum_{k=1}^{N^2} (\sqrt{k+1} - \sqrt{k}) < S_N < \sum_{k=1}^{N^2} (\sqrt{k} - \sqrt{k-1}).$$

The sums on the left and right sides telescope:

$$\sum_{k=1}^{N^2} (\sqrt{k+1} - \sqrt{k}) = \sqrt{N^2+1} - 1 > N - 1,$$

$$\sum_{k=1}^{N^2} (\sqrt{k} - \sqrt{k-1}) = \sqrt{N^2} - \sqrt{0} = N,$$

and from here the result follows.

38. Because of concavity of $\sin x$ in $[0, \frac{\pi}{2}]$ we have $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$, hence

$$\int_0^{\frac{\pi}{2}} e^{\sin x} dx \geq \int_0^{\frac{\pi}{2}} e^{\frac{2}{\pi}x} dx = \frac{\pi}{2} \left[e^{\frac{2}{\pi}x} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}(e - 1).$$

MAIN INEQUALITIES

1. **Arithmetic Mean, Geometric Mean, Harmonic Mean Inequalities.** Let a_1, \dots, a_n be positive numbers. Then the following inequalities hold:

$$\underbrace{\frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}}_{\text{Harmonic Mean}} \leq \underbrace{\sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n}}_{\text{Geometric Mean}} \leq \underbrace{\frac{a_1 + \dots + a_n}{n}}_{\text{Arithmetic Mean}}$$

In all cases equality holds if and only if $a_1 = \dots = a_n$.

2. **Power Means Inequality.** The AM-GM, GM-HM and AM-HM inequalities are particular cases of a more general kind of inequality called *Power Means Inequality*.

Let r be a non-zero real number. We define the r -mean or r th power mean of positive numbers a_1, \dots, a_n as follows:

$$M^r(a_1, \dots, a_n) = \left(\frac{1}{n} \sum_{i=1}^n a_i^r \right)^{1/r}.$$

The ordinary arithmetic mean is M^1 , M^2 is the quadratic mean, M^{-1} is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

$$M^0(a_1, \dots, a_n) = \left(\prod_{i=1}^n a_i \right)^{1/n}.$$

Then, for any real numbers r, s such that $r < s$, the following inequality holds:

$$M^r(a_1, \dots, a_n) \leq M^s(a_1, \dots, a_n).$$

Equality holds if and only if $a_1 = \dots = a_n$.

2.1. *Power Means Sub/Superadditivity.* Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive real numbers.

(a) If $r > 1$, then the r -mean is *subadditive*, i.e.:

$$M^r(a_1 + b_1, \dots, a_n + b_n) \leq M^r(a_1, \dots, a_n) + M^r(b_1, \dots, b_n).$$

(b) If $r < 1$, then the r -mean is *superadditive*, i.e.:

$$M^r(a_1 + b_1, \dots, a_n + b_n) \geq M^r(a_1, \dots, a_n) + M^r(b_1, \dots, b_n).$$

Equality holds if and only if (a_1, \dots, a_n) and (b_1, \dots, b_n) are proportional.

3. **Cauchy-Schwarz.**

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right).$$

4. **Hölder.** If $p > 1$ and $1/p + 1/q = 1$ then

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}.$$

(For $p = q = 2$ we get Cauchy-Schwarz.)

5. **Minkowski.** If $p > 1$ then

$$\left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p \right)^{1/p}.$$

Equality holds iff (a_1, \dots, a_n) and (b_1, \dots, b_n) are proportional.

6. **Norm Monotonicity.** If $a_i > 0$ ($i = 1, \dots, n$), $s > t > 0$, then

$$\left(\sum_{i=1}^n a_i^s \right)^{1/s} \leq \left(\sum_{i=1}^n a_i^t \right)^{1/t}.$$

7. **Chebyshev.** Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be sequences of real numbers which are monotonic in the same direction (we have $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$, or we could reverse all inequalities.) Then

$$\frac{1}{n} \sum_{i=1}^n a_i b_i \geq \left(\frac{1}{n} \sum_{i=1}^n a_i \right) \left(\frac{1}{n} \sum_{i=1}^n b_i \right).$$

8. **Rearrangement Inequality.** For every choice of real numbers $x_1 \leq \dots \leq x_n$ and $y_1 \leq \dots \leq y_n$, and any permutation $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ of x_1, \dots, x_n , we have

$$x_n y_1 + \dots + x_1 y_n \leq x_{\sigma(1)} y_1 + \dots + x_{\sigma(n)} y_n \leq x_1 y_1 + \dots + x_n y_n.$$

If the numbers are different, e.g., $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$, then the lower bound is attained only for the permutation which reverses the order, i.e. $\sigma(i) = n - i + 1$, and the upper bound is attained only for the identity, i.e. $\sigma(i) = i$, for $i = 1, \dots, n$.

9. **Schur.** If x, y, z are positive real numbers and k is a real number such that $k \geq 1$, then

$$x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) \geq 0.$$

For $k = 1$ the inequality becomes

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x).$$

10. Weighted Power Means Inequality. Let w_1, \dots, w_n be positive real numbers such that $w_1 + \dots + w_n = 1$. Let r be a non-zero real number. We define the r th weighted power mean of non-negative numbers a_1, \dots, a_n as follows:

$$M_w^r(a_1, \dots, a_n) = \left(\sum_{i=1}^n w_i a_i^r \right)^{1/r}.$$

As $r \rightarrow 0$ the r th weighted power mean tends to:

$$M_w^0(a_1, \dots, a_n) = \left(\prod_{i=1}^n a_i^{w_i} \right).$$

which we call 0th weighted power mean. If $w_i = 1/n$ we get the ordinary r th power means. Then for any real numbers r, s such that $r < s$, the following inequality holds:

$$M_w^r(a_1, \dots, a_n) \leq M_w^s(a_1, \dots, a_n).$$

11. Convexity. A function $f : (a, b) \rightarrow \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for every $x, y \in (a, b)$, $0 \leq \lambda \leq 1$. Graphically, the condition is that for $x < t < y$ the point $(t, f(t))$ should lie below or on the line connecting the points $(x, f(x))$ and $(y, f(y))$.

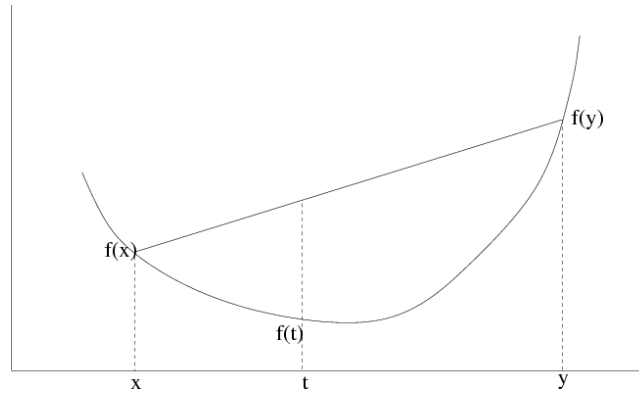


FIGURE 1. Convex function.