PUTNAM TRAINING
RECURRENCES

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REMARK. This is a list of exercises on recurrences. —Miguel A. Lerma

EXERCISES

1. Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$.

2. Determine the maximum number of regions in the plane that are determined by n "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

3. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

4. (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets are selfish.

5. Let $a_1, a_2, \ldots, a_n$ be an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $1, 2, 3, 4$, then $2, 4, 3, 1$ is not a derangement, but $2, 1, 4, 3$ is. Let D_n denote the number of derangements of an n-element sequence. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

6. Let α, β be two (real or complex) numbers, and define the sequence $a_n = \alpha^n + \beta^n$ ($n = 1, 2, 3, \ldots$). Assume that a_1 and a_2 are integers with the same parity (both even or both odd). Prove that a_n is an integer for every $n \geq 1$.

7. Let t_1, t_2, t_3 be integers, and let $\lambda_1, \lambda_2, \lambda_3$ be real or complex numbers. Define the sequence $a_n = \lambda_1 t_1^n + \lambda_2 t_2^n + \lambda_3 t_3^n$ for $n = 0, 1, 2$. Prove that if a_0, a_1, and a_2 are integers then a_n is an integer for every $n \geq 0$.

8. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$, and that the sequence $\{x_n\}$ converges to some real number. Find x_1.

9. (Putnam 2015-A2) Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$.

Find an odd prime factor of a_{2015}.

Hints

1. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n.

2. The $(n + 1)$th "vee" divides the existing regions into how many further regions?

3. The tilings of a $n \times 2$ rectangle by dominoes can be divided into two classes depending on whether we place the rightmost domino vertically or horizontally.

4. The minimal selfish subsets of $\{1, 2, \dots, n\}$ can be divided into two classes depending on whether they contain n or not.

5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. How many possible values can b_n have? Once we have fixed the value of b_n, divide the possible derangements into two appropriate classes.

6. Find a recurrence for a_n.

7. Find a recurrence for a_n.

8. Find a general solution to the recurrence and determine for which value(s) of x_1 the sequence converges.

9. Prove that a_n divides a_{nk} if k is odd.

Solutions

1. Let $f(n)$ be that number. Then we easily find $f(0) = 1$ (the empty subset), $f(1) = 2$ (including the empty subset), $f(2) = 3$, $f(3) = 5$, $f(4) = 8$, ... suggesting that $f(n) = F_{n+2}$ (shifted Fibonacci sequence). We prove this by showing that $f(n)$ verifies the same recurrence as the Fibonacci sequence. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n. The number of the ones not containing n is just $f(n-1)$. On the other hand the ones containing n cannot contain $n-1$, so their number equals $f(n-2)$. Hence $f(n) = f(n-1) + f(n-2)$, QED.

2. Let x_n be the number of regions in the plane determined by n "vee"s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n+1)$th "vee" intersects each of the other "vee"s at 4 points, so it is divided into $4n + 1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n + 1$. So the answer is

$$x_n = 2 + (4 + 1) + (4 \cdot 2 + 1) + \cdots + (4 \cdot (n-1) + 1) = 2n^2 - n + 1\,.$$

3. Let x_n be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in x_{n-1} ways, or we can place two horizontal dominoes to the right and tile the rest in x_{n-2} ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

4. Let f_n denote the number of minimal selfish subsets of $\{1, 2, \dots, n\}$. For $n = 1$ we have that the only selfish set of $\{1\}$ is $\{1\}$, and it is minimal. For $n = 2$ we have two selfish sets, namely $\{1\}$ and $\{1, 2\}$, but only $\{1\}$ is minimal. So $f_1 = 1$ and $f_2 = 1$. For $n > 2$ the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish set containing n, by removing n from the set and subtracting 1 from each remaining element we obtain a minimal selfish subset of $\{1, 2, \dots, n\}$. Conversely, any minimal selfish subset of $\{1, 2, \dots, n-2\}$ gives raise to a minimal selfish subset of $\{1, 2, \dots, n\}$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ containing n is f_{n-2}. Thus $f_n = f_{n-1} + f_{n-2}$, which together with $f_1 = f_2 = 1$ implies that $f_n = F_n$ (nth Fibonacci number.)

5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. The element b_n can be any of $a_1, \dots, a_{n-1}$, so there are $n-1$ possibilities for its value. Once we have fixed the value of $b_n = a_k$ for some $k = 1, \dots, n-1$, the derangement can be of one of two classes: either $b_k = a_n$, or $b_k \neq a_n$. The first class coincides with the derangements of $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{n-1}$, and there are D_{n-2} of them. The second class coincides with the derangements of $a_1, \dots, a_{n-1}$ with a_k replaced with a_n, and there are D_{n-1} of them.

6. Let $s = \alpha + \beta$, $p = \alpha\beta$. We have that $s = a_1$ is an integer. On the other hand we know that a_1 and a_2 have the same parity, hence $a_1^2 - a_2$ is even, and $p = \dfrac{a_1^2 - a_2}{2}$ is an integer. Note also that

$$\alpha^{n+2} + \beta^{n+2} = (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta(\alpha^n + \beta^n)\,,$$

i.e.,

$$a_{n+2} = s\,a_{n+1} - p\,a_n\,,$$

and we get the desired result by induction.

7. Consider the polynomial

$$p(x) = (x - t_1)(x - t_2)(x - t_3) = x^3 - c_1x^2 - c_2x - c_3\,,$$

where $c_1 = t_1 + t_2 + t_3$, $c_2 = -t_1t_2 - t_1t_3 - t_2t_3$, $c_3 = t_1t_2t_3$. Note that since t_1, t_2, t_3 are integers, then c_1, c_2, c_3 are also integers. Also, the t_i's are roots of p, i.e., $p(t_i) = 0$, hence $t_i^3 = c_1t_i^2 + c_2t_i + c_3$ $(i = 1, 2, 3)$. Multiplying by t_i^n we get $t_i^{n+3} = c_1t_i^{n+2} + c_2t_i^{n+1} + c_3t_i^n$. By definition $a_n = \lambda_1t_1^n + \lambda_2t_2^n + \lambda_3t_3^n$, so by linearity:

$$a_{n+3} = c_1a_{n+2} + c_2a_{n+1} + c_3a_n \qquad (n = 0, 1, 2, \dots)\,.$$

Finally, given that a_0, a_1, a_2 are integers by hypothesis, and that c_1, c_2, c_3 are also integers, the result follows by induction.

8. The general solution for the recurrence can be expressed using the roots of its characteristic polynomial

$$x^2 - \frac{10x}{3} + 1 = 0\,.$$

The roots are 3 and 1/3, hence a general solution is $x_n = A \cdot 3^n + B \cdot 3^{-n}$. If the sequence converges then $A = 0$, and the condition $x_0 = 18$ yields $B = 18$, hence the sequence is $x_n = 18 \cdot 3^{-n}$, the limit is 0, and $x_1 = 18/3 = 6$.

9. First we prove that a_n divides a_{nk} if k is odd. One way to do it is by solving the recurrence explicitly. Its characteristic polynomial is $x^2 - 4x + 1$, with roots $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3} = \alpha^{-1}$, so that $a_n = A\alpha^n + B\alpha^{-n}$. Using $a_0 = 1$, $a_1 = 2$ we get $A = B = 1/2$, hence $a_n = \frac{1}{2}(\alpha^n + \alpha^{-n})$.
We next use the identity $x^k + y^k = (x + y)(x^{k-1} - x^{k-2}y + \cdots + y^{k-1})$, valid when k is odd, replacing $x = \alpha^n$, and $y = \alpha^{-n}$, and grouping terms of the form $\alpha^j + \alpha^{-j}$:

$$\begin{aligned}
a_{nk} &= \frac{1}{2}(\alpha^{nk} + \alpha^{-nk}) \\
&= \frac{1}{2}(\alpha^n + \alpha^{-n}))(\alpha^{n(k-1)} - \alpha^{n(k-3)} + \cdots + \alpha^{n(1-k)}) \\
&= \frac{1}{2}(\alpha^n + \beta^n)\{(\alpha^{n(k-1)} + \alpha^{n(1-k)}) - (\alpha^{n(k-3)}\alpha^{n(3-k)}) + \cdots + 1\} \\
&= a_n(2a_{n(k-1)} - 2a_{n(k-3)} + \cdots + 1)\,.
\end{aligned}$$

This proves that in fact a_n divides a_{nk} for k odd.

An alternative way to do it without solving the recurrence explicitly is to look at the sequence modulo a positive integer $m \geq 2$. First notice that we can extend the recurrence backwards using $a_{n-2} = 4a_{n-1} - a_n$. Since $a_{-1} = 2 = a_1$ the sequence turns out to be symmetric: $a_{-n} = a_n$ for every n. On the other hand, if $a_n \equiv 0 \pmod{m}$, then $a_{n+1} \equiv 4a_n - a_{n-1} \equiv -a_{n-1} \pmod{m}$, and from here (working the recurrence forward and backwards), $a_{n+j} \equiv -a_{n-j} \equiv -a_{j-n} \pmod{m}$ for every j. In particular for $j = 2in$ we get $a_{n(2i+1)} \equiv -a_{n(2i-1)} \pmod{m}$, and from here (by induction on i) we get $a_{n(2i+1)} \equiv 0 \pmod{m}$. Letting $m = a_n$ we get $a_{n(2i+1)} \equiv 0 \pmod{a_n}$, i.e., a_n divides $a_{n(2i+1)}$.

Finally we notice that $2015 = 5 \cdot 403$, hence a_5 divides a_{2015}. We have $a_5 = 362 = 2 \cdot 181$, so 181 is a prime dividing a_{2015}.