Northwestern University

Math 220 Final Exam
Fall Quarter 2018
December 12, 2018

Last name: SOLUTIONS ______ Email address: ______

First name: ______ NetID: ______

Instructions

- Mark your instructor's name.

  ____ Cañez

  ____ Chu

  ____ Frankel

  ____ Porod

  ____ Richter

  ____ Wyman

- This examination consists of 15 pages, not including this cover page. Verify that your copy of this examination contains all 15 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 10 questions for a total of 200 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (20 points) Suppose $f(x)$ is a function satisfying $f'(x) = 2e^x + \frac{3}{1+x^2} - x^{1/3}$ and $f(0) = 3$. Find $f(x)$.

$$f(x) = 2e^x + 3\arctan(x) - \frac{x^{4/3}}{4/3} + c$$

$$f(0) = 2\cdot e^0 + 3\cdot\arctan(0) - 0 + c = 3$$

$$2 + \underbrace{3\cdot 0}_{0} + c = 3$$

$$c = 1$$

$$\boxed{f(x) = 2e^x + 3\arctan(x) - \frac{3x^{4/3}}{4} + 1}$$

2. (This problem has **three** parts.) Determine the value of each of the following limits.

(a) (5 points) $\lim_{x\to 3} \frac{\sin x - x^3}{1+e^x}$

$$\lim_{x\to 3} \frac{\sin x - x^3}{1+e^x} = \boxed{\frac{\sin(3)-27}{1+e^3}}$$

(b) (5 points) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = \lim_{x\to 2} (x+3) = \boxed{5}$$

(c) (5 points) $\lim_{x\to 0} \frac{1-\cos x}{x}$

$$\lim_{x\to 0} \frac{1-\cos x}{x} \overset{\text{L'Hôspital}}{=} \lim_{x\to 0} \frac{\sin x}{1} = \boxed{0}$$

$$\left[\frac{0}{0}\right]$$

3. (This problem has **six** parts.) Let $f(x) = \dfrac{x}{x^2+1}$.

(a) (2 points) Find the x- and y-intercepts of the graph of f.

x-intercepts: Set $y=0$ $\quad 0 = \dfrac{x}{x^2+1}$

$\longrightarrow$ $x=0$ is the only x-intercept

y-intercept: Set $x=0$

$$y = \frac{0}{0^2+1} = 0$$

$y=0$ is the y-intercept

(b) (5 points) Find all horizontal and vertical asymptotes of f, if any.

$f(x) = \dfrac{x}{x^2+1}$ is continuous for all x in $\mathbb{R}$. Therefore, f has <u>no</u> vertical asymptote.

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \underbrace{\lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}}}_{\Downarrow} = 0$$

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \qquad = 0$$

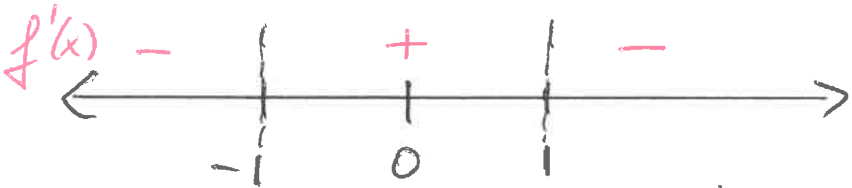
$y=0$ is a horizontal asymptote. (It is the only horizontal asymptote for f.)

Recall that $f(x) = \frac{x}{x^2+1}$.

(c) (5 points) On which intervals is f increasing? Decreasing?

$$f'(x) = \frac{1\cdot(x^2+1) - 2x\cdot x}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = 0 \longrightarrow x^2-1=0$$

$$\underbrace{x = \pm 1}_{\text{critical points}}$$



$f'(-2) < 0$ $\quad f'(0) > 0$ $\quad f'(2) < 0$

f is increasing on $(-1, 1)$
f is decreasing on $(-\infty, -1)$ and on $(1, \infty)$.

(d) (5 points) Locate the local maxima and minima of f.

The critical point $x=1$ is a local maximum by the 1st derivative test and the result from part (c).

The critical point $x=-1$ is a local minimum by the 1st derivative test and the result from part (c).

(e) (5 points) Find all inflection points and determine the intervals where the graph of f is concave upward or concave downward. It might help to know that the second derivative of f is $f''(x) = \frac{2x^3-6x}{(1+x^2)^3}$.

Set $f''(x) = \frac{2x^3-6x}{(1+x^2)^3} = 0 \longrightarrow 2x(x^2-3)=0$

$x=0, \quad x = \pm\sqrt{3}$

$-$ | $+$ | $-$ | $+$ $f''(x)$

$-\sqrt{3}$ $\quad -1 \quad$ 0 $\quad 1 \quad$ $\sqrt{3}$

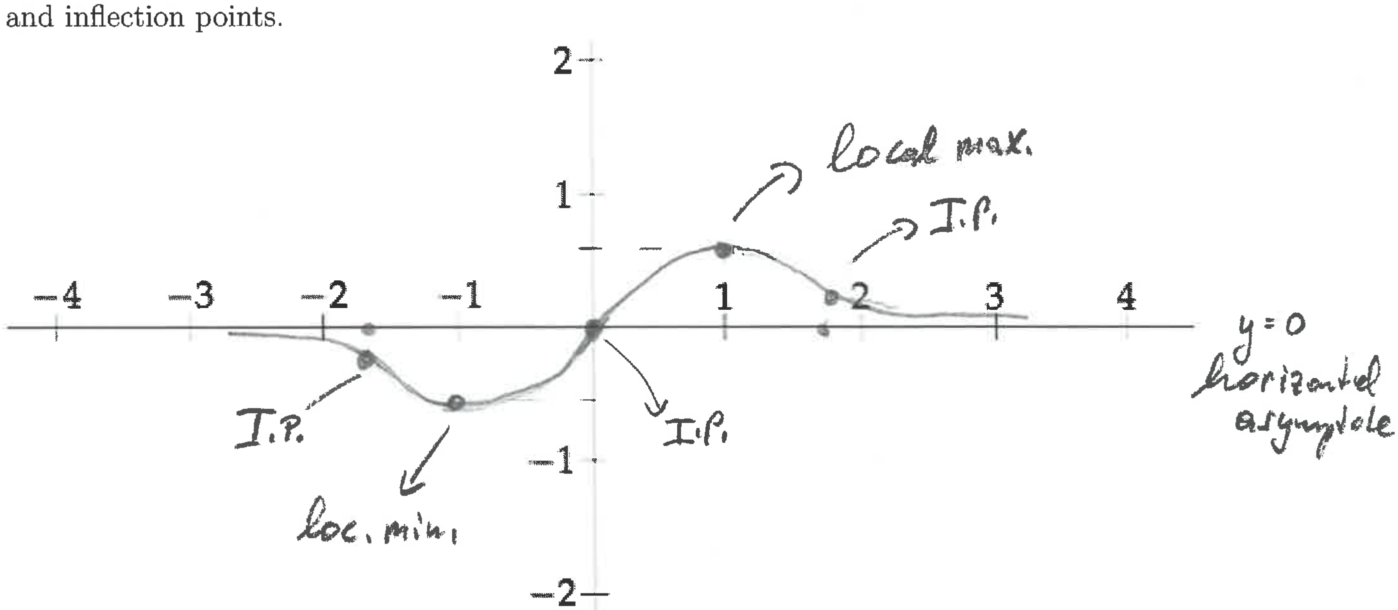
$f''(-2)<0$ $\quad f''(-1)>0$ $\quad f''(1)<0$ $\quad f''(2)>0$

$\longrightarrow$ f is concave upward on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$.

f is concave downward on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$.

Inflection points: $0, \sqrt{3}, -\sqrt{3}$.

(f) (3 points) Sketch the graph of $f(x)$. Be sure to label any local maxima, minima, intercepts, asymptotes, and inflection points.



4. (20 points) Find the absolute maximum and minimum values of the function

$$f(x) = -3x^4 + 4x^3 + 12x^2 + 1$$

over the interval $[-2, 1]$, and the points at which they occur.

Closed interval method:

$$f'(x) = -12x^3 + 12x^2 + 24x = 0$$

$$-12x(x^2 - x - 2) = 0$$

$$-12x(x-2)(x+1) = 0$$

$x = 0$

$\cancel{x = 2}$ → lies outside the interval $[-2, 1]$.

$x = -1$

$f(-2) = -3 \cdot 16 - 4 \cdot 8 + 12 \cdot 4 + 1 = \boxed{-31}$ → absolute minimum at $x = -2$

$f(1) = -3 \cdot 1 + 4 \cdot 1 + 12 \cdot 1 + 1 = \boxed{14}$ → absolute maximum at $x = 1$

$f(0) = -3 \cdot 0 + 4 \cdot 0 + 12 \cdot 0 + 1 = 1$

$f(-1) = -3 \cdot 1 - 4 \cdot 1 + 12 \cdot 1 + 1 = 6$

5. (20 points) Suppose f is a differentiable function satisfying $f(0) = 0$ and

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 2.$$

Set

$$g(x) = xf(xe^x) + f(\sin(f(x))).$$

Find $g'(0)$.

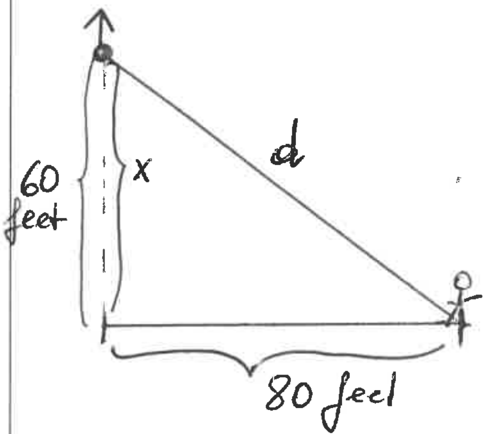
We know : $f'(0) = 2$, $f(0) = 0$

$$g'(x) = f(xe^x) + \underbrace{xf'(xe^x) \cdot [e^x + xe^x]} + f'(\sin(f(x))) \cdot \cos(f(x)) \cdot f'(x)$$

$$g'(0) = f(0) + 0 \cdot (\cdots\cdots) + f'(0) \cdot \cos(0) \cdot f'(0)$$

$$= 0 + 0 + 2 \cdot 1 \cdot 2 = \boxed{4}$$

6. (20 points) A balloon is rising at a constant speed of 4 ft/sec. A child is watching the rising balloon from a spot on the ground that is 80 feet away from the launch pad of the balloon. How fast is the distance between the child and balloon increasing when the balloon is at a height of 60 feet?



$x'(t) = 4$ ft/sec

$d^2 = \sqrt{x^2 + 80^2}$

$d'(t) = ?$ when $x = 60$.

$$d(t) = \sqrt{[x(t)]^2 + 6400}$$

$$d'(t) = \frac{2\cdot x(t)\cdot x'(t)}{2\sqrt{[x(t)]^2+6400}} = \frac{\cancel{2}\cdot 60\cdot 4}{\cancel{2}\sqrt{3600+6400}} = \frac{24\cancel{0}}{10\cancel{0}} = \frac{12}{5}$$

Answer: The distance between the child and the balloon is increasing at a rate of $\frac{12}{5}$ ft/sec when the height of the balloon is 60 feet.

7. Let $f(x) = (x+1)^{\frac{1}{x}}$.

(a) (10 points) Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}(x+1)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(x+1)} = e^{\lim_{x\to\infty}\frac{\ln(x+1)}{x}}$$

this is O.K. because the exp. fct. is continuous.

$$\lim_{x\to\infty}\frac{\ln(x+1)}{x} \overset{\text{L'Hôspital}}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{1} = 0$$

$\left[\frac{\infty}{\infty}\right]$

Answer: $\lim_{x\to\infty}(x+1)^{\frac{1}{x}} = e^0 = \boxed{1}$

(b) (10 points) Find $f'(x)$.

$$y = (x+1)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(x+1)$$

$$\frac{y'}{y} = \left(-\frac{1}{x^2}\right)\ln(x+1) + \frac{1}{x(x+1)} \longrightarrow$$

$$y' = f'(x) = \boxed{(x+1)^{\frac{1}{x}}\left[\left(-\frac{1}{x^2}\right)\ln(x+1) + \frac{1}{x(x+1)}\right]}$$

8. Let $f(x) = e^{\frac{1}{x^2+2x+1}}$ with domain $(-1, \infty)$.

(a) (5 points) Explain how you know $f(x)$ is a strictly decreasing function.

$$\boxed{f'(x) = e^{\frac{1}{(x+1)^2}} \cdot \left(\frac{-2}{(x+1)^3}\right) < 0}$$ for all x in $(-1,\infty)$

Since
$$\left.\begin{array}{l} e^{\frac{1}{(x+1)^2}} > 0 \\ -2 < 0 \\ x+1 > 0 \text{ and } (x+1)^3 > 0 \end{array}\right\}$$ for all x in $(-1,\infty)$.

Since $f'(x) < 0$, $f(x)$ is strictly decreasing

(b) (5 points) Determine the range of $f(x)$. [*Hint:* To answer this question, you first need to compute two limits.]

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} e^{\frac{1}{(x+1)^2}} = e^{\lim_{x\to -1^+} \frac{1}{(x+1)^2}} = e^{\infty} = \infty$$

O.K since the exp. fct. is continuous

$$\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} \frac{1}{(x+1)^2}} = e^0 = 1$$

Altogether, since f is strictly decreasing and continuous: $\boxed{\text{range}(f) = (1,\infty)}$

(c) (10 points) Find $(f^{-1})'(e)$.

Note that $f(0) = e$. Therefore, $f^{-1}(e) = 0$

Use: $[f^{-1}]'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$[f^{-1}]'(e) = \frac{1}{f'(0)} = \left[e^{\frac{1}{(0+1)^2}} \cdot \frac{-2}{(0+1)^3}\right]^{-1} = \boxed{\frac{1}{-2e}}$$

9. (20 points) Explain why there is exactly one number satisfying the equation

$$e^x = 2 - 30x.$$

Be sure to name any theorems you use, and to explain why they are applicable.

Let $f(x) = e^x - 2 + 30x$

$f'(x) = e^x + 30 > 0$ for all x in $\mathbb{R}$.

→ $f(x)$ is a strictly increasing function, therefore one-to-one.

$$\left.\begin{array}{l} f(-1) = e^{-1} - 2 - 30 < 0 \\ f(1) = e^1 - 2 + 30 > 0 \end{array}\right\} \longrightarrow$$

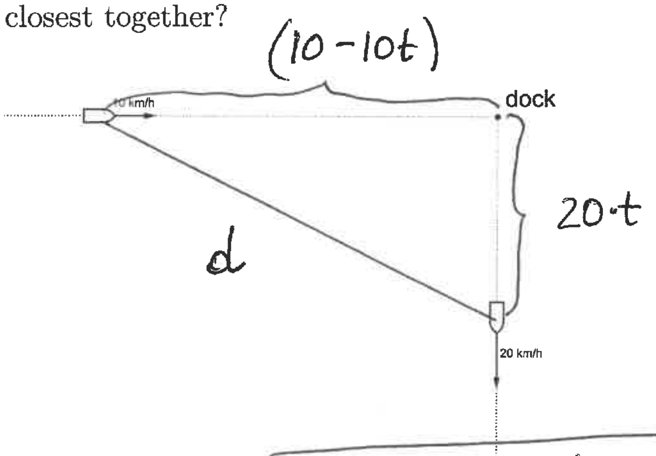
Since f is a continuous function, we can apply the Intermediate Value Theorem to f and the interval $[-1, 1]$.

So there exists a value c in $(-1, 1)$ such that $f(c) = 0$. This implies $e^c = 2 - 30c$.

Since f is one-to-one, there cannot be a different value b such that $f(b) = 0$.

Therefore, there exists exactly one value c such that $e^c = 2 - 30c$.

10. (20 points) A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 10 km/h and reaches the same dock at 3:00 PM. At what time are the two boats closest together?



t ... in hours

$$d(t) = \sqrt{(10-10t)^2 + (20t)^2}$$

$\longrightarrow$ When is this function a minimum for t in $[0,1]$?

Closed interval method:

$$d'(t) = \frac{\cancel{2}(10-10t)(-10) + \cancel{2}\cdot 20t \cdot 20}{\cancel{2}\sqrt{(10-10t)^2+(20t)^2}} = 0 \longrightarrow$$

$$-100 + 100t + 400t = 0$$
$$500t = 100$$
$$t = 1/5$$

critical point for $d(t)$.

$d(0) = 10$ km

$d(1) = 20$ km

$d\left(\frac{1}{5}\right) = \sqrt{8^2 + 4^2} = \sqrt{80} \approx \boxed{8.94 \text{ km}}$

Absolute Minimum at $t = \frac{1}{5}$.

Answer: The two boats are closest together at 2:12 pm.

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.