Northwestern University

Math 220 Midterm 1
Fall Quarter 2018
October 23, 2018

Last name: SOLUTIONS Email address: ________

First name: ________ NetID: ________

Instructions

- Mark your instructor's name.

  ____ Cañez

  ____ Chu

  ____ Frankel

  ____ Porod

  ____ Richter

  ____ Wyman

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**, and circle your choice. You do not need to justify your answers.

   (a) (3 points) The limit $\lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{|x|}\right)$ exists and equals 0.

   TRUE **FALSE**

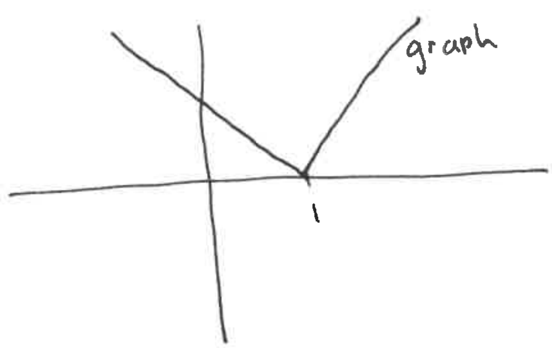
   For $x<0$, $|x|=-x$

   So $\lim\limits_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{|x|}\right) = \lim\limits_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{-x}\right)$

   $= \lim\limits_{x\to 0^-} 0 = 0$

   (b) (3 points) The function $f(x)=2|x-1|$ is differentiable at 1.

   TRUE **FALSE**



"Sharp" point at $x=1$ means it is not differentiable

2. (10 points) Determine the value of c for which the following function is continuous at every point of $(-\infty, \infty)$.

$$f(x) = \begin{cases} x - c & x \leq 1 \\ \frac{x^2-1}{x-1} + c & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - c) = 1 - c$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{x^2-1}{x-1} + c \right)$$

$$= \lim_{x \to 1^+} \left(\frac{(x-1)(x+1)}{x-1} + c \right)$$

$$= \lim_{x \to 1^+} (x+1+c) = 2 + c$$

For $\lim_{x \to 1} f(x)$ to exist need these limits to be equal: $1 - c = 2 + c$ So $c = -1/2$.

To be continuous at $x = 1$ need

$$\lim_{x \to 1} f(x) = f(1) = 1 - (-1/2) = 3/2$$

which is true.

3. (10 points) Explain why there is a number between 3 and 7 satisfying the following equation. Name any theorem you use in your explanation, and be sure to mention why it applies.

$$2(x-1)\sin\left(\frac{3\pi}{x-1}\right) = x$$

Set $f(x) = 2(x-1)\sin\left(\frac{3\pi}{x-1}\right) - x$. This is continuous on $[3,7]$ since $2(x-1)$, $\sin\left(\frac{3\pi}{x-1}\right)$, and x are all continuous on $[3,7]$ and $f(x)$ is built by multiplying and subtracting these continuous functions.

Now, $f(3) = 2(2)\sin\left(\frac{3\pi}{2}\right) - 3 = -4-3 = -7$

$f(7) = 2(6)\sin\left(\frac{3\pi}{6}\right) - 7 = 12-7 = 5$

$f(3) < 0 < f(7)$ so Intermediate Value Theorem (applies since f is continuous) implies that there is some x between 3 and 7 satisfying $f(x) = 2(x-1)\sin\left(\frac{3\pi}{x-1}\right) - x = 0$

4. This problem has **two** parts.

(a) (4 points) Write the **limit definition** of the derivative of a function $f(x)$ at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) (12 points) Use the limit definition of the derivative to compute $f'(2)$ where f is the function defined by

$$f(x) = \frac{7x-1}{x+11}.$$

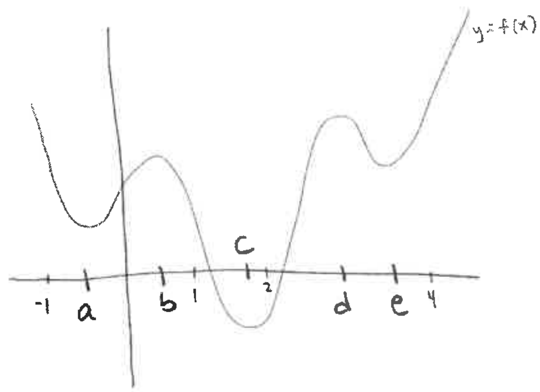
$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{7(2+h)-1}{(2+h)+11} - \cancel{\frac{13}{13}}\,1}{h}$$

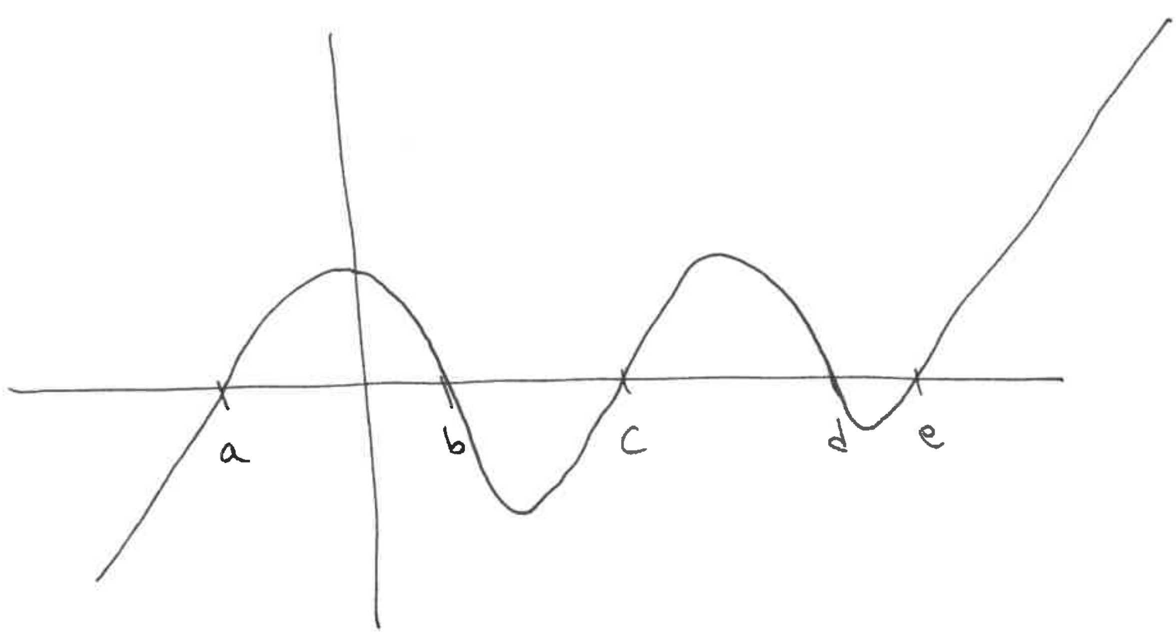
$$= \lim_{h \to 0} \frac{\frac{14+7h-1-(h+13)}{(h+13)}}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h(h+13)} = \lim_{h \to 0} \frac{6}{h+13} = \frac{6}{13}$$

5. This problem has **two** parts; the second is on the next page. Below is the graph of a function f:



(a) (10 points) Using the symbols a, b, c, d, and e, label the points in the graph of f at which f' is zero, and draw a rough sketch of the graph of f'.



(b) (10 points) Determine whether the derivative of each the following functions at the indicated point is positive, negative, or zero. The function $f(x)$ is still the one whose graph is drawn on the previous page.

$f(x)^5$ at $x = 1$ $\qquad$ $xf(x)$ at $x = -1$ $\qquad$ $f(x^2)$ at $x = 2$

$$\frac{d}{dx} f(x)^5 = 5 f(x)^4 f'(x)$$

@ $x=1$: $5 \underbrace{f(1)^4}_{\text{pos}} \underbrace{f'(1)}_{\text{neg}}$ so negative

$$\frac{d}{dx} x f(x) = f(x) + x f'(x)$$

@ $x=-1$: $\underbrace{f(-1)}_{\text{pos}} + \underbrace{(-1)}_{\text{neg}} \underbrace{f'(-1)}_{\text{neg}}$ so positive

$$\frac{d}{dx} f(x^2) = f'(x^2)\, 2x$$

@ $x=2$: $\underbrace{f'(4)}_{\text{pos}} 4$ so positive

6. Compute the derivative of each of the following functions. You do NOT have the simplify your answer.

(a) (9 points) $f(x) = [\sin(x) + (5+x)^3]^4$

$$f'(x) = 4\left[\sin x + (5+x)^3\right]^3 \left(\cos x + 3(5+x)^2\right)$$

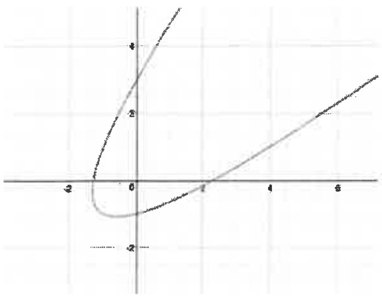
$x^{1/3}$

(b) (9 points) $g(x) = \frac{\cos(x)}{x^2+1} + \sin(x)\sqrt{x^3}$

$$g'(x) = \frac{(x^2+1)(-\sin x) - \cos x\,(2x)}{(x^2+1)^2}$$

$$+ \cos x \sqrt{x^3} + \sin x \left(\frac{1}{3} x^{-2/3}\right)$$

7. (10 points) Find the point P on the curve $x^2 - 2xy + y^2 = x + 2y + 3$ where the tangent line to the curve has slope $-\frac{1}{2}$. For reference, here is a picture of the curve:



differentiate both sides:

$$2x - 2y - 2xy' + 2yy' = 1 + 2y'$$

so $2x - 2y - 1 = 2xy' - 2yy' + 2y'$

$= (2x - 2y + 2)y'$

so $y' = \dfrac{2x - 2y - 1}{2x - 2y + 2}$

$y' = -\frac{1}{2}$ when $-2(2x - 2y - 1) = 2x - 2y + 2$

$-4x + 4y + 2 = 2x - 2y + 2$

$6y = 6x$ so $\boxed{x = y}$

Plug into curve: $x^2 - 2x^2 + x^2 = x + 2x + 3$

$0 = 3x + 3$ so $x = -1$

$y = x = -1$

Point is $(-1, -1)$, which is on the curve

8. (10 points) A spherical balloon is inflated with helium at a rate of $100\pi \frac{\text{ft}^3}{\text{min}}$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing at that instant? (Recall that the volume of a balloon of radius r is $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$.)

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

$$100\pi = \frac{12\pi}{3}(5)^2 \frac{dr}{dt} \longrightarrow \frac{dr}{dt} = \frac{100\pi}{4\cdot 25\pi} = 1 \text{ ft/min}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 4\pi 2r \frac{dr}{dt} = 8\pi(5)(1) = 40\pi \text{ ft}^2/\text{min}$$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

DO NOT WRITE ON THIS PAGE.