Northwestern University

Math 220 Midterm 2
Fall Quarter 2018
November 13, 2018

Last name: SOLUTIONS Email address: ________

First name: ________ NetID: ________

Instructions

- Mark your instructor's name.

  ____ Cañez
  ____ Chu
  ____ Frankel
  ____ Porod
  ____ Richter
  ____ Wyman

- This examination consists of 9 pages, not including this cover page. Verify that your copy of this examination contains all 9 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 7 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**, and circle your choice. You do not need to justify your answers.

(a) (3 points) If $f(x)$ is a one-to-one function satisfying $f'(x) > 0$ for all x, then the inverse function $f^{-1}(x)$ is increasing.

TRUE **FALSE**

$(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ is always positive since the denominator is always positive

So f^{-1} is increasing

(b) (3 points) The limit $\lim_{x \to -\infty} \arctan x$ does not exist. (Recall that the notations $\arctan x$ and $\tan^{-1} x$ mean the same thing.)

TRUE **FALSE**

$\lim_{x \to -\infty} \arctan x = -\pi/2$ since $y = \arctan x$ has a horizontal asymptote at $y = -\pi/2$

2. Let $f(x) = \sqrt[5]{x^2 + 7}$.

(a) (6 points) Find the linearization to $f(x)$ at $a = 5$.

$$f(x) = (x^2+7)^{1/5}$$

$$f'(x) = \frac{1}{5}(x^2+7)^{-4/5}(2x)$$

linearization is

$$L(x) = f(5) + f'(5)(x-5)$$

$$= \sqrt[5]{32} + \frac{1}{5(32)^{4/5}}(10)(x-5)$$

$$= 2 + \frac{10}{5 \cdot 2^4}(x-5) = 2 + \frac{1}{8}(x-5)$$

(b) (6 points) Find an approximation to the value of $\sqrt[5]{43}$. (Part (a) is relevant.)

$$\sqrt[5]{43} = f(6)$$

So $\sqrt[5]{43} \approx L(6) = f(5) + f'(5)(6-5)$

$$= 2 + \frac{1}{8}(1) = 2 + \frac{1}{8}$$

3. Consider the function $h(x) = \frac{2^x}{2^x - 8}$.

(a) (6 points) Find all horizontal asymptotes of $h(x)$.

$$\lim_{x\to\infty} \frac{2^x}{2^x-8}\cdot\frac{\frac{1}{2^x}}{\frac{1}{2^x}} = \lim_{x\to\infty}\frac{1}{1-\frac{8}{2^x}} = \frac{1}{1-0} = 1$$

$$\lim_{x\to-\infty}\frac{2^x}{2^x-8} = \frac{0}{0-8} = 0$$

So $h(x)$ has horizontal asymptotes at $y=1$ and $y=0$

(b) (10 points) Compute the limits $\lim_{x\to 3^-} h(x)$ and $\lim_{x\to 3^+} h(x)$.

$$\lim_{x\to 3^-}\frac{2^x}{2^x-8}$$

numerator approaches 8

denominator approaches 0 from negative direction

since $2^x < 8$, so limit is $-\infty$

$$\lim_{x\to 3^+}\frac{2^x}{2^x-8}$$

numerator approaches 8

denominator approaches 0 from positive direction

since $2^x > 8$, so limit is ∞

(c) (3 points) Find all vertical asymptotes of $h(x)$.

Since $\lim_{x\to 3^-} h(x) = -\infty$ and $\lim_{x\to 3^+} h(x) = \infty$,

$h(x)$ has vertical asymptote at $x=3$

4. Let $f(x) = \ln(x+1) - \ln x$.

(a) (6 points) Compute $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} [\ln(x+1) - \ln x] = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right)$$

Compute $\lim_{x\to\infty} \frac{x+1}{x} \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1} = 1$

By continuity,

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left[\lim_{x\to\infty} \frac{x+1}{x}\right]$$

$$= \ln 1 = 0$$

(b) (6 points) Find the inverse function f^{-1}.

$$y = \ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right)$$

Take $e^{\text{both sides}}$: $e^y = e^{\ln\left(\frac{x+1}{x}\right)} = \frac{x+1}{x}$

$$xe^y = x+1 \rightarrow xe^y - x = 1$$

$$x(e^y - 1) = 1$$

$$x = \frac{1}{e^y - 1}$$

So $f^{-1}(x) = \frac{1}{e^x - 1}$

5. (15 points) Find the derivative of $f(x) = (\cos x)^x$

Set $y = (\cos x)^x$

So $\ln y = \ln (\cos x)^x$

$= x \ln(\cos x)$

Take derivatives:

$$\frac{1}{y} y' = \ln(\cos x) + x \frac{1}{\cos x} (\sin x)$$

$$= \ln(\cos x) + x \tan x$$

$$y' = y \left[\ln(\cos x) + x \tan x \right]$$

$$= (\cos x)^x \left[\ln(\cos x) + x \tan x \right]$$

6. Compute the derivatives of the following functions.

(a) (9 points) $f(x) = \arcsin(e^{-x})$

$$f'(x) = \frac{1}{\sqrt{1-(e^{-x})^2}} \left(-e^{-x}\right)$$

derivative of e^{-x}

arcsin derivative

$$= \frac{1}{\sqrt{1-e^{-2x}}} \left(-e^{-x}\right)$$

(b) (9 points) $g(x) = \ln(x^2 2^x)$

$$g(x) = \ln(x^2) + \ln(2^x)$$

$$= 2\ln x + x \ln 2$$

$$g'(x) = \frac{2}{x} + \ln 2$$

or $\quad g'(x) = \frac{1}{x^2 2^x}\left(2x \cdot 2^x + x^2 2^x \ln 2\right)$

$$= \frac{2x \cdot 2^x}{x^2 \cdot 2^x} + \frac{x^2 2^x \ln 2}{x^2 2^x} = \frac{2}{x} + \ln 2$$

7. Compute the following limits.

(a) (9 points) $\lim_{x\to\infty} \frac{x^2-5x}{2^x}$

Both numerator and denominator go to ∞,

So can apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{x^2-5x}{2^x} = \lim_{x\to\infty} \frac{2x-5}{2^x \ln 2}$$ still indeterminate of type $\frac{\infty}{\infty}$!

So apply L'Hospital again:

$$\lim_{x\to\infty} \frac{2x-5}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0$$

(b) (9 points) $\lim_{x\to\infty} (1+x)^{\frac{1}{x}}$

$$\lim_{x\to\infty} (1+x)^{1/x} = \lim_{x\to\infty} e^{\ln (1+x)^{1/x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(1+x)}$$

Compute: $\lim_{x\to\infty} \frac{\ln(1+x)}{x}$ indeterminate of type $\frac{\infty}{\infty}$

Apply L'Hospital: $= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0$

So

$$\lim_{x\to\infty} (1+x)^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(1+x)} = e^0 = 1$$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

DO NOT WRITE ON THIS PAGE.