# Northwestern University

Math 224 Final Exam
Fall Quarter 2017
December 6, 2017

Last name: SOLUTIONS ____________ Email address: ____________

First name: ____________ NetID: ____________

Instructions

- This examination consists of 17 pages, not including this cover page. Verify that your copy of this examination contains all 17 pages. If your examination is missing any pages, then obtain a new copy immediately.
- This examination consists of 10 questions for a total of 200 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**, and circle your choice. You do not need to justify your answers.

(a) (3 points) If the sum $\sum_{n=0}^{\infty} c_n 5^n$ converges, then $\sum_{n=0}^{\infty} (-1)^n c_n 2^n$ converges as well.

5 is in interval of convergence of $\sum_{n=0}^{\infty} c_n x^n$

so radius at least 5, meaning $-2$ is in there as well

**TRUE** **FALSE**

(b) (3 points) If $T_4(x)$ is the 4-th degree Taylor polynomial of a function $f(x)$ centered at $a = 1$, then $f(x) - T_4(x) = \frac{f^{(5)}(0)}{5!}(x-1)^5$.

for some $c$ between $x$ and $0$, but $c$ not $0$ itself

$f(x) - T_4(x) = \frac{f^{(5)}(c)}{5!}(x-1)^5$

**TRUE** **FALSE**

(c) (3 points) If $\int_{1/2}^{\sqrt{2}/2} f(x)\,dx = -2$ then $\int_{\pi/6}^{\pi/4} f(\sin x)\cos x\,dx = -2$.

$u = \sin x$

$du = \cos x\,dx$

$= \int_{\sin \pi/6}^{\sin \pi/4} f(u)\,du$

$= \int_{1/2}^{\sqrt{2}/2} f(u)\,du = -2$

**TRUE** **FALSE**

(d) (3 points) If $a_n \geq 0$ for all $n$ and the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} \frac{a_n}{n}$ also converges.

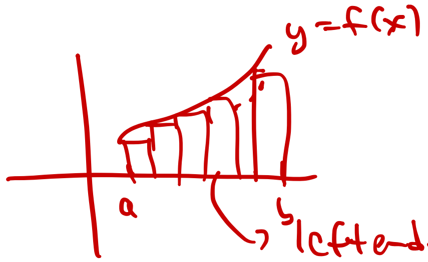
$\frac{a_n}{n} \leq a_n$ so

$\sum a_n$ converges by comparison test

**TRUE** **FALSE**

(e) (3 points) If $f(x)$ is a continuous function on $[a, b]$, the left endpoint Riemann sum approximation for $\int_a^b f(x)\,dx$ for a given partition is larger than the right endpoint Riemann sum approximation for that same partition.

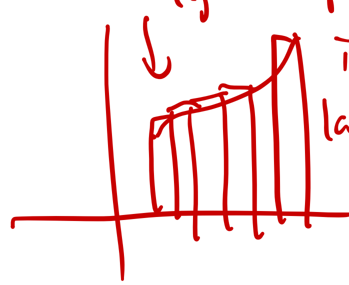
For




left endpoint Riemann sum

**TRUE** **FALSE**

right endpoint is larger



2. (20 points) It is known that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. Use this information to evaluate the following integral.

$$\int_0^\infty x^2 e^{-x^2} dx$$

**Hint:** It may help to rewrite the integrand as $x(xe^{-x^2})$.

$$\int_0^\infty x^2 e^{-x^2} dx = \lim_{b\to\infty} \int_0^b x\, x e^{-x^2} dx$$

$u = x \qquad v = -\frac{1}{2} e^{-x^2}$

$du = dx \qquad dv = xe^{-x^2} dx$

substitution

$t = -x^2, \; dt = -2x\,dx$

$-\frac{1}{2} dt = x\,dx$

$\int xe^{-x^2} dx = -\frac{1}{2}\int e^t dt$

$$\int_0^b x\, xe^{-x^2} dx = -\frac{1}{2} x e^{-x^2}\Big|_0^b + \frac{1}{2}\int_0^b e^{-x^2} dx$$

$$= -\frac{1}{2}\frac{b}{e^{b^2}} + \frac{1}{2}\int_0^b e^{-x^2} dx$$

Take $\lim_{b\to\infty}$:

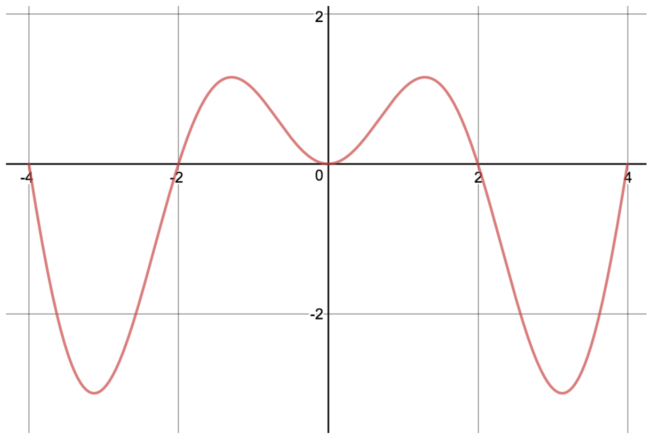
$$\int_0^\infty x^2 e^{-x^2} dx = \lim_{b\to\infty} -\frac{1}{2}\frac{b}{e^{b^2}} + \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

L'Hopital

$$= \lim_{b\to\infty} -\frac{1}{2}\frac{1}{2be^{b^2}} + \frac{1}{2}\frac{\sqrt{\pi}}{2} = 0 + \sqrt{\pi}/4$$

$$= \boxed{\sqrt{\pi}/4}$$

3. (17 points) The following is the graph of a function $g(x)$ on the interval $[-4.4]$.




Consider the function $f(x)$ on the interval $[e^{-4}, e^{4}]$ defined by

$$f(x) = \int_0^{\ln x} g(t)dt.$$

Find all $x$ in the interval $[e^{-4}, e^{4}]$ satisfying $f'(x) = 0$.

Set $G(x) = \int_0^x g(t)dt$. So $G'(x) = g(x)$ by Fundamental Theorem of Calculus

$f(x) = G(\ln x) - G(0)$

$f'(x) = G'(\ln x)(\ln x)'$ by Chain rule

$= g(\ln x)\frac{1}{x}$, which is zero when

$g(\ln x) = 0$

from graph need $\ln x = -4, 2, 0, 2, 4$

so $\boxed{x = e^{-4}, e^{2}, e^{0}, e^{2}, e^{4}}$

4. (20 points) Evaluate the following integral.

$$\int \frac{1}{\sqrt{x^2 - 2x + 5}}\,dx$$

**Hint:** Find values of $a, b$ satisfying $x^2 - 2x + 5 = (x - a)^2 + b^2$.

$$x^2 - 2x + 5 = (x-a)^2 + b^2 = x^2 - 2ax + a^2 + b^2$$

Need $-2 = -2a$ and $5 = a^2 + b^2$

$\hookrightarrow a = 1$ $\quad 5 = 1 + b^2 \rightarrow b = 2$

$$\int \frac{1}{\sqrt{x^2-2x+5}}\,dx = \int \frac{1}{\sqrt{(x-1)^2+4}}\,dx$$

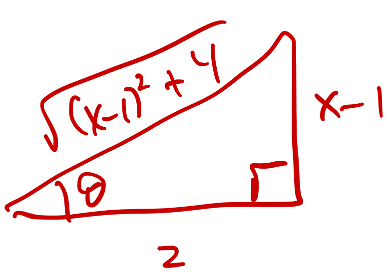
$x - 1 = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1}{\sqrt{4\tan^2\theta + 4}}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{2\sec\theta}\,2\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$\frac{x-1}{2} = \tan\theta$




$$= \ln|\sec\theta + \tan\theta| + C$$

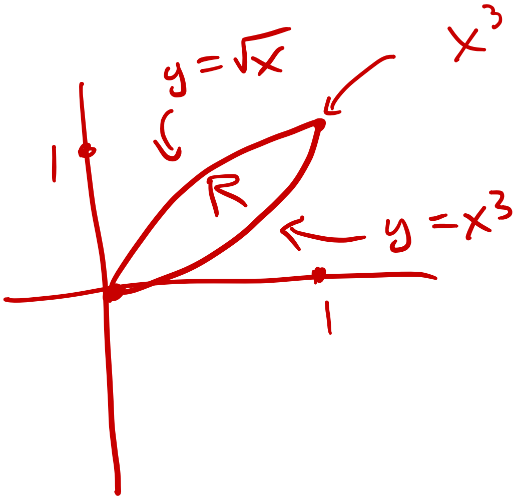
$$= \boxed{\ln\left|\frac{\sqrt{(x-1)^2+4}}{2} + \frac{x-1}{2}\right| + C}$$

5. Let $R$ be the region in the first quadrant of the $xy$-plane bounded by the curves $y = x^3$ and $y = \sqrt{x}$.

(a) (10 points) Find the area of $R$.

$x^3 = \sqrt{x} \Rightarrow x^6 = x \Rightarrow x(x^5 - 1) = 0$

so $x = 0$ or $x = 1$

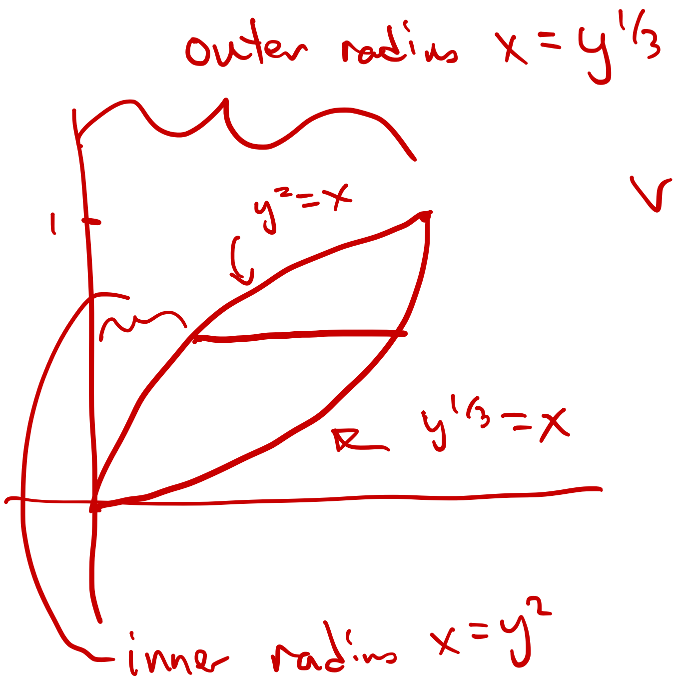



$$\text{area}(R) = \int_0^1 (\sqrt{x} - x^3)\,dx$$

$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right)\Big|_0^1$$

$$= \boxed{\frac{2}{3} - \frac{1}{4}}$$

(b) (10 points) Set up an integral which computes the volume of the solid obtained by revolving $R$ about the $y$-axis. You do **not** have to evaluate this integral.




volume

$$= \int_0^1 \left[\pi \left(y^{1/3}\right)^2 - \pi \left(y^2\right)^2\right] dy$$

6. This problem has two parts; the second is on the next page.

(a) (12 points) Determine whether the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n^2+n-2}$$

converges or diverges. Be clear about which convergence test you are using.

Compare to $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n^2+n-2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\sqrt{(n+1)n^3}}{\sqrt{(n^2+n-2)^2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^4+n}{n^4+2n^3-3n^2-4n+4}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1+1/n^3}{1+2/n-3/n^2-4/n^3+4/n^4}}$$

$$= \sqrt{\frac{1}{1}} = 1 > 0 \text{ and finite}$$

So since $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test, $\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n^2+n-2}$ converges by limit comparison test.

(b) (12 points) Determine whether the improper integral

$$\int_{\frac{1}{2}}^{1} \frac{1}{x(x-1)}dx$$

converges or diverges. If it converges, find its value.

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Longrightarrow 1 = A(x-1) + Bx$$

$$x = 1 \rightarrow B = 1$$

$$x = 0 \rightarrow A = -1$$

$$\int_{1/2}^{1} \frac{1}{x(x-1)}dx = \int_{1/2}^{1} \left(-\frac{1}{x} + \frac{1}{x-1}\right)dx$$

$$= \lim_{b \to 1} \int_{1/2}^{b} \left(-\frac{1}{x} + \frac{1}{x-1}\right)dx$$

$$= \lim_{b \to 1} \left(-\underbrace{\cancel{\ln|b|}}_{\ln 1 = 0} + \ln\frac{1}{2} + \underbrace{\ln|b-1|}_{\text{diverges to } -\infty} - \ln\frac{1}{2}\right)$$

So limit goes to $-\infty$

So $\int_{1/2}^{1} \frac{1}{x(x-1)}dx$ diverges.

7. This problem has two parts; the second is on the next page.

(a) (12 points) Find the Maclaurin series of the function $f(x) = x\ln(1+x^2)$ by somehow manipulating the series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$.

integrate with respect to $y$:

$$-\ln|1-y| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} + C$$

Set $y=0$ to get $-\ln 1 = 0 + C$ so $\underline{\underline{C=0}}$.

multiply by $-1$: $\ln|1-y| = \sum_{n=0}^{\infty} -\frac{y^{n+1}}{n+1}$

set $y=-x^2$: $\ln(1+x^2) = \sum_{n=0}^{\infty} -\frac{(-x^2)^{n+1}}{n+1}$

Note: $(-1)^{n+2} = (-1)^n$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$$

multiply by $x$: $x\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}\cdot x}{n+1}$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{n+1}}$$

(b) (10 points) Determine the value of the following infinite sum. You must also give a reason as to why the sum actually converges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{3n+3}}{2^{2n+3}(n+1)}$$

**Hint:** Evaluate the series from part (a) at an appropriate value of $x$.

Set $x = -\frac{1}{2}$ in $\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{n+1}$

to get $\ln\left(1+\frac{1}{4}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(-\frac{1}{2}\right)^{2n+3}}{n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+3}}{2^{2n+3}(n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{3n+3}}{2^{2n+3}(n+1)}$$

Since this series is obtained by integrating the geometric series, which has radius of convergence 1, this series does too, so $\left|-\frac{1}{2}\right| < 1$ and $-1/2$ is within interval of convergence

8. (a) (10 points) Suppose $f(x)$ is a function satisfying $f^{(n)}(-1) = \dfrac{(n+3)!}{3^n}$ for each $n \geq 0$. Find the Taylor series of $f(x)$ centered at $a = -1$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{\frac{(n+3)!}{3^n}}{n!}(x+1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{3^n}(x+1)^n$$

Since $\dfrac{(n+3)!}{n!} = (n+3)(n+2)(n+1)$

(b) (10 points) Find a single function $f(x)$ satisfying all of the following conditions:

$$f(0) = 3, \quad f'(0) = -11, \quad f''(0) = 9, \quad f'''(0) = 5, \quad f^{(4)}(0) = -3, \quad f^{(9)}(0) = 22$$

Take Taylor series with all other derivatives equal to 0 to get a polynomial:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(9)}(0)}{9!}x^9$$

$$= \boxed{3 - 11x + \frac{9}{2}x^2 + \frac{5}{6}x^3 - \frac{3}{24}x^4 + \frac{22}{9!}x^9}$$

9. This problem has two parts; the second is on the next page.

(a) (10 points) Find the Maclaurin series of the following function.

$$f(x) = x(\cos x - 1)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$\cos x - 1$ gets rid of $1$, so

$$\cos x - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Multiply by $x$:

$$x(\cos x - 1) = \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}}.$$

(b) (12 points) Find a bound on the error obtained when approximating $f(x)$ using its third degree Taylor polynomial $T_3(x)$ centered at 0 on the interval $[-\frac{1}{10}, \frac{1}{10}]$. Here, $f(x)$ is the same function as in part (a).

$$|f(x) - T_3(x)| = \left|\frac{f^{(4)}(c)}{4!}x^4\right|$$ for some $c$ between $x$ and $0$

$f(x) = x\cos x - x$

$f'(x) = \cos x - x\sin x - 1$

$f''(x) = -2\sin x - x\cos x$

$f'''(x) = -3\cos x + x\sin x$

$f^{(4)}(x) = 4\sin x + x\cos x$

bounds $4|\sin c|$

bounds $|c||\cos c|$ for $|c| \le \frac{1}{10}$

So $|f^{(4)}(c)| = |4\sin c + c\cos c| \le 4 + 1 = 5$

Thus error $= \left|\frac{f^{(4)}(c)}{4!}x^4\right| \le \frac{5}{4!}\left(\frac{1}{10}\right)^4$

interval of convergence is $(-3, 1]$



10. (20 points) Determine the interval of convergence of the following power series.

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{4^n(n+2)}(x+1)^n \quad a_n$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}(x+1)^{n+1}}{4^{n+1}(n+3)} \right| \left| \frac{4^n(n+2)}{(-2)^n(x+1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{2}{4}|x+1|\left(\frac{n+2}{n+3}\right) = \frac{1}{2}\lim_{n\to\infty} |x+1|\left(\frac{1+2/n}{1+3/n}\right)$$

$$= \frac{1}{2}|x+1| .$$

According to ratio test, to converge need $\frac{1}{2}|x+1| < 1$

So $|x+1| < 2$

which gives $x$ in $(-1-2, -1+2)$

$(-3, 1)$

Test endpoints:

$\underline{x=-3}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{4^n(n+2)}(-2)^n = \sum_{n=0}^{\infty} \frac{1}{n+2}$ which diverges by comparison with $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$

$\underline{x=1}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{4^n(n+2)}2^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$, $\displaystyle\lim_{n\to\infty}\frac{1}{n+2} = 0$ and

$b_n = \frac{1}{n+2}$ decreases since denominator increases, so converges by alternating series test.