Northwestern University

Math 224 Midterm 1
Fall Quarter 2017
October 16, 2017

Last name: SOLUTIONS ____________ Email address: ____________

First name: ____________ NetID: ____________

Instructions

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

The following identities may be helpful:

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \sec^2 x = \tan^2 x + 1$$

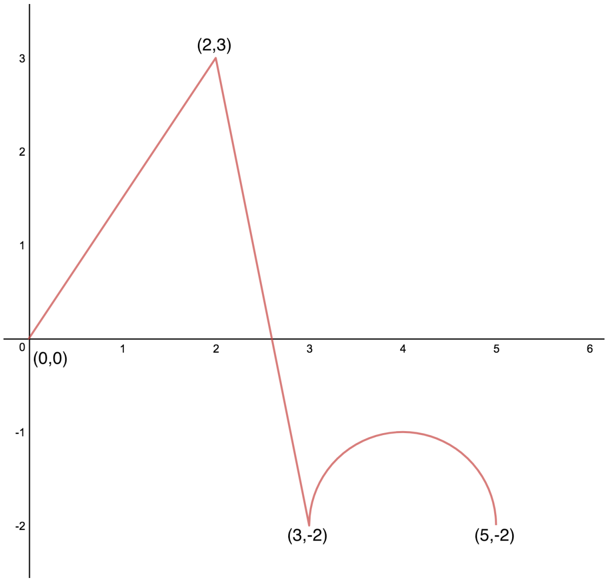
$$\int \tan x\,dx = \ln|\sec x| + C \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

1. This question has five parts; the rest are on the next page. Determine whether each of the following statements is **TRUE** or **FALSE**, and circle your choice. You do not need to justify your answers.

   (a) (3 points) Consider the following graph of a function $g(t)$.



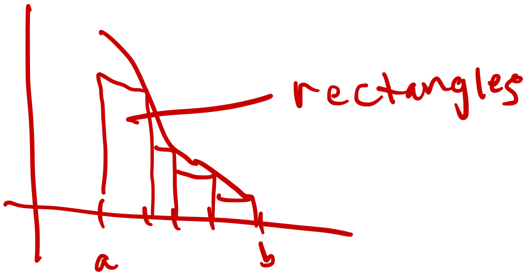
   The function $G(x) = \int_0^x g(t)\,dt$ has a maximum at $x = 2$.

   $G'(2) = g(2) = 3$ is not even zero

   TRUE [✓ **FALSE**]

   (b) (3 points) Let f be a decreasing function on an interval $[a, b]$. If R is the value of a Riemann sum for f defined using right endpoints as sample points, then $R \leq \int_a^b f(x)\,dx$.

   [✓ **TRUE**] **FALSE**



under approximate area

(c) (3 points) If f and g are continuous functions on an interval $[a, b]$, then the identity

$$\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

always holds.

$\int_0^1 xe^x\,dx$ is not

TRUE [✓ FALSE]

$\int_0^1 x\,dx \int_0^1 e^x\,dx$

for instance

(d) (3 points) If $\int_1^e f(x)\,dx = 4$, then $\int_0^1 e^x f(e^x)\,dx = 4$.

$u = e^x$ $du = e^x dx$

[✓ TRUE] FALSE

$\int_0^1 e^x f(e^x) dx = \int_1^e f(u)\,du = 4$

(e) (3 points) If f is continuous and $F(x) = \int_1^{x^2} f(t)\,dt$, then $F'(x) = f(x^2)$.

TRUE [✓ FALSE]

$F'(x) = f(x^2) \cdot (x^2)'$

$= f(x^2)\,2x$

2. (a) (8 points) Find a function $f(x)$ and an interval $[a, b]$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\cos^2\left(2+\frac{i}{n}\right)+\left(2+\frac{i}{n}\right)\right)$$

equals the value of $\int_a^b f(x)\,dx$.

$2+\frac{i}{n}$ gives $2+\frac{1}{n}, 2+\frac{2}{n}, \dots, 2+\frac{n}{n}=3$

So interval is $[2,3]$. $\frac{1}{n}$ is width of subinterval

So function is $\cos^2 x + x$

(b) (8 points) Find the value of this limit by evaluating the integral found in part (a). You do not have to simplify your answer.

$$\int_2^3 (\cos^2 x + x)\,dx$$

$$= \int_2^3 \cos^2 x\,dx + \int_2^3 x\,dx$$

$$= \frac{1}{2}\int_2^3 (1+\cos 2x)\,dx + \frac{1}{2}x^2\Big|_2^3$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)\Big|_2^3 + \frac{1}{2}x^2\Big|_2^3$$

$$= \boxed{\frac{1}{2}\left(3+\frac{1}{2}\sin 6 - 2 - \frac{1}{2}\sin 2\right) + \frac{1}{2}(9-4)}$$

3. This question has two parts; the second is on the next page. Evaluate the following indefinite integrals.

   (a) (12 points) $\int \frac{e^{2x}}{e^{4x} + e^{2x}}\, dx$

$u = e^{2x}$

$du = 2e^{2x}\, dx$

$$\int \frac{e^{2x}}{e^{4x}+e^{2x}}\, dx = \int \frac{\frac{1}{2}\, du}{u^2+u}$$

partial fractions: $\frac{1}{u^2+u} = \frac{A}{u+1} + \frac{B}{u} \longrightarrow 1 = Au + B(u+1)$

$u=0: \quad 1 = B$

$u=-1: \quad 1 = -A$

$$\frac{1}{2}\int \frac{1}{u^2+u}\, du = \frac{1}{2}\int\left(-\frac{1}{u+1} + \frac{1}{u}\right) du$$

$$= \frac{1}{2}\left(-\ln|u+1| + \ln|u|\right) + C$$

$$= \boxed{\frac{1}{2}\left(-\ln|e^{2x}+1| + \ln|e^{2x}|\right) + C}$$

(b) (12 points) $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

$x = 2\cos\theta \qquad dx = -2\sin\theta\, d\theta$

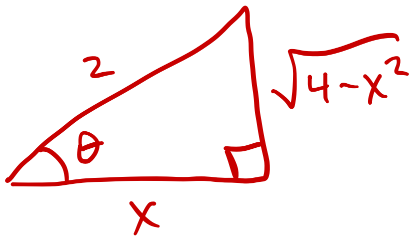
$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}\,(-2\sin\theta)\,d\theta$$

$$= \int \frac{-2\sin\theta}{4\cos^2\theta\sqrt{4\sin^2\theta}}\,d\theta$$

$$= -\frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta = -\frac{1}{4}\int \sec^2\theta\, d\theta$$

$\frac{x}{2} = \cos\theta$

$$= -\frac{1}{4}\tan\theta + C$$



$$= \boxed{-\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C}$$

4. An oven thermometer has the following readings over a 30 minute period

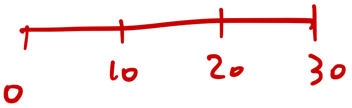
t	0	5	10	15	20	25	30
$T(t)$	347	350	361	365	370	369	372

where $T(t)$ is the temperature at time t. Estimate the average temperature over this time in the following two ways:

(a) (8 points) Use the midpoint rule, for $n = 3$ intervals. You do not have to simplify your answer.

$$\frac{1}{30}\int_0^{30} T(t)\,dt$$

↳



$\Delta x = 10$

$$\approx \frac{1}{30}\left(T(5)\Delta x + T(15)\Delta x + T(25)\Delta x\right)$$

$$= \frac{1}{30}\left(350\cdot 10 + 365\cdot 10 + 369\cdot 10\right)$$

(b) (8 points) Use Simpson's rule, for $n = 6$ intervals. You do not have to simplify your answer.

↳

$\Delta x = 5$

$$\frac{1}{30}\int_0^{30} T(t)\,dt$$

$$\approx \frac{1}{30}\,\frac{\Delta x}{3}\Big(T(0) + 4T(5) + 2T(10) + 4T(15)$$
$$+ 2T(20) + 4T(25) + T(30)\Big)$$

$$= \frac{5}{90}\Big(347 + 4\cdot 350 + 2\cdot 361 + 4\cdot 365 + 2\cdot 370$$
$$+ 4\cdot 369 + 372\Big)$$

5. (15 points) Suppose a function f is differentiable and satisfies $f(6) = \frac{1}{3}$, $f(2) = \frac{1}{2}$, and

$$\int_2^6 f(x)\,dx = -1.$$

Find the value of

$$\int_4^{36} f'(\sqrt{x})\,dx.$$

Hint: Try the substitution $t = \sqrt{x}$.

$t = \sqrt{x}$ $\quad dt = \frac{1}{2\sqrt{x}} dx \longrightarrow dx = 2\sqrt{x}\, dt$

$= 2t\, dt$

$$\int_4^{36} f'(\sqrt{x})\,dx = \int_2^6 f'(t)\, 2t\, dt$$

$u = 2t \qquad v = f(t)$

$du = 2dt \qquad dv = f'(t)\,dt$

$$\int_2^6 f'(t) 2t\, dt = 2t\, f(t)\Big|_2^6 - \int_2^6 2 f(t)\, dt$$

$$= 12 f(6) - 4 f(2) - 2\int_2^6 f(t) dt$$

$$= 12 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} - 2(-1)$$

$$= \boxed{4}$$

6. (14 points) Evaluate the definite integral

$$\int_0^1 x \arctan x \, dx.$$

Hint: At some point, you may need to use polynomial long division.

$u = \arctan x$ $\qquad$ $v = \frac{1}{2}x^2$

$du = \frac{1}{1+x^2}\,dx$ $\qquad$ $dv = x\,dx$

from front page

$$\int_0^1 x \arctan x \, dx = \frac{1}{2}x^2 \arctan x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$\begin{array}{r|l} & 1 \\ \hline x^2+1 & x^2 \\ & -(x^2+1) \\ \hline & -1 \end{array}$$

so $\displaystyle\int_0^1 \frac{x^2}{1+x^2}\,dx = \int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx$

$$= (x - \arctan x)\Big|_0^1$$

$$= 1 - \arctan 1 - (0 - \arctan 0)$$

$$= 1 - \pi/4$$

$$= \frac{1}{2}\arctan 1 - \frac{1}{2}(1 - \pi/4)$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{2}(1 - \pi/4)}$$

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