Northwestern University

Math 224 Midterm II
Fall Quarter 2017
November 13, 2017

Last name: SOLUTIONS ______ Email address: ______

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Instructions

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**, and circle your choice. You do not need to justify your answers.

   (a) (3 points) If the sequence b_n consists only of positive terms and the series $\sum_{n=1}^{\infty}(-b_n)$ converges, then $\sum_{n=1}^{\infty}(-b_n)$ converges absolutely.

   $\sum |-b_n| = \sum b_n = -\sum(-b_n)$ converges

   TRUE **FALSE**

   (b) (3 points) The integral $\int_{-1}^{0} \frac{1}{x^3}\,dx$ converges.

   $\int_{-1}^{0} \frac{1}{x^3}dx = \lim_{b\to 0} \int_{-1}^{b} \frac{1}{x^3}dx$

   $= \lim_{b\to 0} -\frac{1}{x^2}\Big|_{-1}^{b}$

   $= \lim_{b\to \infty} \left(-\frac{1}{b^2}+1\right) = -\infty$

   TRUE **FALSE**

   (c) (3 points) Suppose a series $\sum_{n=1}^{\infty} a_n$ has terms $a_n = f(n)$ given by the values at n of a function f which is positive, decreasing, and continuous. If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$.

   integral = area under curve

   series = sum of areas of rectangles.

   TRUE **FALSE**

   no reason why series and integral must have same value

   (d) (3 points) The arclength of a curve $y = f(x)$, for values of x in an interval $[a, b]$, is always less than or equal to the value of $\int_a^b f(x)\,dx$.

   arclength of $y = 1/2$ from $x=0$ to $x=1$ is 1, but area under is $\frac{1}{2}\cdot 1 = 1/2$

   TRUE **FALSE**

   (e) (3 points) The series $\sum_{n=1}^{\infty} e^{\arctan n}$ converges.

   TRUE **FALSE**

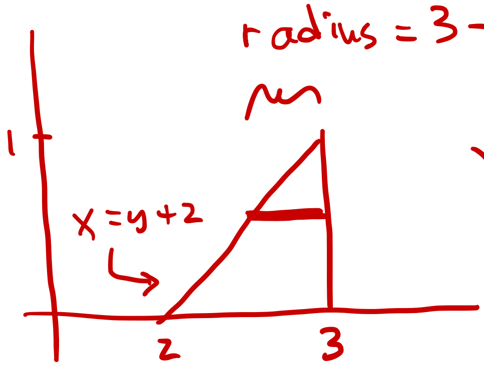
   $\lim_{n\to\infty} e^{\arctan n} = e^{\lim_{n\to\infty} \arctan n} = e^{\pi/2} \neq 0$

   So series diverges by test for divergence

2. Let R be the triangle in the first quadrant bounded by the lines $y = x - 2$, $y = 0$, and $x = 3$.

(a) (8 points) Setup an integral which gives the volume of the solid obtained by revolving R around the line $x = 3$ and compute this volume.

radius $= 3-(y+2) = 1-y$



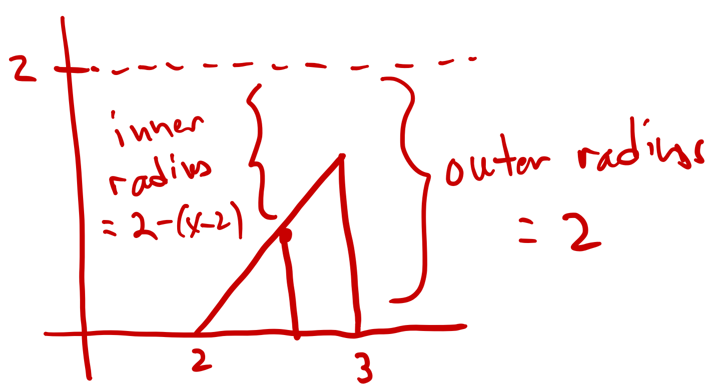
$$\text{volume} = \int_0^1 \pi(1-y)^2\,dy$$

$$= \pi\int_0^1 (1-2y+y^2)\,dy$$

$$= \pi\left(y - y^2 + \tfrac{1}{3}y^3\right)\Big|_0^1$$

$$= \boxed{\pi/3}$$

(b) (8 points) Setup an integral which gives the volume of the solid obtained by revolving R around the line $y = 2$. Do not evaluate this integral.



$$\text{volume} = \int_2^3 \left[\pi(\text{outer})^2 - \pi(\text{inner})^2\right]dx$$

$$= \int_2^3 \left[\pi(2)^2 - \pi(4-x)^2\right]dx$$

3. This question has **two** parts; the second is on the next page. Do the following series converge or diverge? Be clear about which convergence test you are using.

   (a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3 + 2\sin n}{n}$

Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by the p-series test

$$\frac{3+2\sin n}{n} \geq \frac{3-2}{n} = \frac{1}{n}$$

So since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\displaystyle\sum_{n=1}^{\infty} \frac{3+2\sin n}{n}$ diverges by the comparison test

(b) (10 points) $\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{\ln n}+\frac{1}{n}\right)$

$b_n = \frac{1}{\ln n} + \frac{1}{n}$ consists of positive terms

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{1}{\ln n} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{\ln n} + \lim_{n\to\infty}\frac{1}{n} = 0 + 0 = 0$$

where each limit is 0 since denominators go to ∞ and numerators remain constant

$f(x) = \frac{1}{\ln x} + \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x(\ln x)^2} - \frac{1}{x^2}$ is negative for $x \geq 2$

So b_n decreases.

So $\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{\ln n}+\frac{1}{n}\right)$ converges by the alternating series test

4. (15 points) Use the integral test to determine if the series

$$\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$$

converges or diverges. Be sure to justify the hypotheses needed in order to apply the integral test!

$f(x) = \frac{\ln x}{x^2}$ is continuous and positive for $x \geq 3$

$$f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \ln x \ (2x)}{(x^2)^2} = \frac{x(1-2\ln x)}{x^4} < 0$$

since $1 - 2\ln x < 0$ for $x \geq 3$

So integral test applies.

$u = \ln x \qquad v = -\frac{1}{x}$

$du = \frac{1}{x}dx \qquad dv = \frac{1}{x^2}dx$

$$\int_3^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty} \int_3^b \frac{\ln x}{x^2}dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{x}\ln x \Big|_3^b + \int_3^b \frac{1}{x^2}dx \right]$$

$$= \lim_{b\to\infty}\left[-\cancel{\frac{\ln b}{b}}^{\to 0} + \frac{\ln 3}{3} - \cancel{\frac{1}{b}}^{\to 0} + \frac{1}{3}\right] = \frac{\ln 3 + 1}{3}$$

Aside:

$$\lim_{b\to\infty} \frac{\ln b}{b} = \lim_{b\to\infty} \frac{1/b}{1} = 0$$

L'Hopital's Rule

$\int_3^\infty \frac{\ln x}{x^2}dx$ converges so $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges by the integral test

5. This question has **two** parts; the second is on the next page. Do the following series converge or diverge? If they converge, find their value. Be clear about which convergence test you are using.

(a) (12 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 3^{n+2}}{5^{n-1}}$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n 3^n 3^2}{5^n 5^{-1}} = \frac{3^2}{5^{-1}} \sum_{n=2}^{\infty} \left(-\frac{3}{5}\right)^n$$

a geometric series, which converges since $-1 < -\frac{3}{5} < 1$.

$$\text{value} = \boxed{\frac{3^2}{5^{-1}} \underbrace{\left(\frac{1}{1-\left(-\frac{3}{5}\right)} - 1 - \left(-\frac{3}{5}\right) \right)}_{\sum_{n=2}^{\infty} \left(-\frac{3}{5}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{5}\right)^n - 0^{th} \text{ term} - 1^{st} \text{ term}}}$$

(b) (12 points) $\sum_{n=1}^{\infty} \frac{(3n)!}{(-2)^{5n}}$

$$a_n = \frac{(3n)!}{(-2)^{5n}}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(3(n+1))!}{|(-2)^{5(n+1)}|} \cdot \frac{|(-2)^{5n}|}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)!}{2^{5n+5}} \cdot \frac{2^{5n}}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{2^5} = \infty$$

So $\sum_{n=1}^{\infty} \frac{(3n)!}{(-2)^{5n}}$ diverges by the ratio test.

6. (10 points) Find the arclength of the portion of the curve

$$x = (2y)^{\frac{3}{2}} - 9$$

going from the point $(-1, 2)$ to the point $(55, 8)$.

$$x' = \frac{3}{2}(2y)^{1/2}(2) = 3\sqrt{2y}$$

$$\text{arclength} = \int_2^8 \sqrt{1+(x')^2}\, dy$$

$$= \int_2^8 \sqrt{1+18y}\, dy \qquad u = 1+18y$$

$$du = 18\, dy$$

$$\frac{1}{18}\, du = dy$$

$$= \int_{37}^{145} \frac{1}{18} u^{1/2}\, du$$

$$= \frac{1}{18} \frac{2}{3} u^{3/2} \Big|_{37}^{145}$$

$$= \boxed{\frac{1}{27}\left((145)^{3/2} - (37)^{3/2}\right)}$$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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DO NOT WRITE ON THIS PAGE.