Northwestern University

Math 226 Final Exam
Fall Quarter 2019
December 13, 2019

Name: SOLUTIONS

Initials:

netID:

Instructions

- This examination consists of 10 pages, not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- Enter your initials in the indicated box on each page, and enter your Name and netID on the indicated boxes on the cover sheet.
- This examination consists of 8 questions for a total of 100 points.
- You have 2 hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly. Cross out any work that you do not wish to have scored.
- Show all of your work and thoroughly explain your reasoning. Unsupported answers may not earn credit.

Scoring

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	7	
6	13	
7	20	
8	10	
Total:	100	

1. Determine, with justification, whether the given series converges or diverges.

(a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(e^n)}{n^3+1}$

$$0 \le \frac{2+\sin(e^n)}{n^3+1} \le \frac{2+1}{n^3}$$

since $\sum \frac{3}{n^3}$ converges by p-series test, $\sum \frac{2+\sin(e^n)}{n^3+1}$ converges

by direct comparison test

(b) (10 points) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n\ln n}$

$f(x) = \frac{1}{x\ln x}$ is continuous and positive

also decreasing since denominator increases

$$\int_5^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_5^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_5^b$$

$$= \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 5)\right] \text{ diverges}$$

So $\sum \frac{1}{n\ln n}$ diverges by integral test

2. (10 points) Determine the value of the following infinite sum. You do not have to justify the fact that this converges, and your answer can be left in unsimplified form.

$$\sum_{n=4}^{\infty}\left(\frac{3^{n-1}}{5^{n+2}}-\frac{2\cdot 4^n}{3\cdot 7^{n-2}}\right)$$

$$\sum_{n=4}^{\infty}\frac{3^n 3^{-1}}{5^n 5^2}-\sum_{n=4}^{\infty}\frac{2\cdot 4^n}{3\cdot 7^n\cdot 7^{-2}}$$

$$=\frac{3^{-1}}{5^2}\sum_{n=4}^{\infty}\left(\frac{3}{5}\right)^n-\frac{2}{3\cdot 7^{-2}}\sum_{n=4}^{\infty}\left(\frac{4}{7}\right)^n$$

$$=\frac{3^{-1}}{5^2}\left[\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n-\left(\frac{3}{5}\right)^0-\left(\frac{3}{5}\right)^1-\left(\frac{3}{5}\right)^2-\left(\frac{3}{5}\right)^3\right]$$

$$-\frac{2}{3\cdot 7^{-2}}\left[\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n-\left(\frac{4}{7}\right)^0-\left(\frac{4}{7}\right)^1-\left(\frac{4}{7}\right)^2-\left(\frac{4}{7}\right)^3\right]$$

$$=\frac{3^{-1}}{5^2}\left[\frac{1}{1-\frac{3}{5}}-1-\frac{3}{5}-\frac{3^2}{5^2}-\frac{3^3}{5^3}\right]$$

$$-\frac{2}{3\cdot 7^{-2}}\left[\frac{1}{1-\frac{4}{7}}-1-\frac{4}{7}-\frac{4^2}{7^2}-\frac{4^3}{7^3}\right]$$

3. (10 points) Find the interval of convergence of the following power series.

$$\sum_{n=0}^{\infty} \frac{3^{n-1}}{n^2+1}(x-3)^n.$$

$$\lim_{n\to\infty} \frac{\frac{3^n}{(n+1)^2+1}|x-3|^{n+1}}{\frac{3^{n-1}}{n^2+1}|x-3|^n} = \lim_{n\to\infty} 3\left(\frac{n^2+1}{n^2+2n+2}\right)|x-3|$$

$$= 3|x-3| \lim_{n\to\infty} \frac{1+1/n^2}{1+2/n+2/n^2} = 3|x-3|$$

by ratio test need $3|x-3|<1 \Rightarrow |x-3|<1/3$

so at least $\left(3-\frac{1}{3}, 3+\frac{1}{3}\right) = \left(\frac{8}{3}, \frac{10}{3}\right)$

$\underline{x=\frac{8}{3}}$ $\sum_{n=0}^{\infty} \frac{3^{n-1}}{n^2+1}\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3(n^2+1)}$ converges by alternating series test

$\underline{x=\frac{10}{3}}$ $\sum_{n=0}^{\infty} \frac{3^{n-1}}{n^2+1}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3(n^2+1)}$

converges by limit comparison with $\sum \frac{1}{n^2}$.

interval is $\left[\frac{8}{3}, \frac{10}{3}\right]$

4. (10 points) Find a power series representation of the function $f(x) = \frac{x}{(1-2x)^3}$ and determine its radius of convergence. (Just the radius, not the full interval of convergence.) Hint: Start by differentiating $\frac{1}{1-y}$ twice with respect to y.

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

$$\frac{d}{dy}$$

$$\frac{1}{(1-y)^2} = \sum_{n=1}^{\infty} n y^{n-1}$$

$$\frac{d}{dy}$$

$$\frac{2}{(1-y)^3} = \sum_{n=2}^{\infty} n(n-1) y^{n-2}$$

Set $y = 2x$:

$$\frac{2}{(1-2x)^3} = \sum_{n=2}^{\infty} n(n-1) 2^{n-2} x^{n-2}$$

divide 2, multiply x:

$$\frac{x}{(1-2x)^3} = \sum_{n=2}^{\infty} n(n-1) 2^{n-3} x^{n-1}$$

$\sum y^n$ converges for $|y|<1$, so derivatives do as well.

So final series (with $y=2x$) converges for $|2x|<1$ $\Rightarrow |x| < \frac{1}{2}$

radius of convergence is $1/2$

5. (7 points) Find the Taylor series of the function $f(x) = \frac{1}{x}$ centered at 5.

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3} \qquad f'''(x) = -\frac{3\cdot 2}{x^4}$$

$$f^{(4)}(x) = \frac{4\cdot 3\cdot 2}{x^5}$$

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}, \quad f^{(n)}(5) = (-1)^n \frac{n!}{5^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{n!}{5^{n+1}\,n!}(x-5)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{5^{n+1}}$$

6. (13 points) Justify the following equality

$$\int_0^1 \sin(x^3)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(6n+4)}$$

and determine the partial sum of the series on the right with the fewest number of terms which approximates the integral on the left to within an error of $\frac{1}{100{,}000}$.

Hint: Start by finding a power series representation of $\sin(x^3)$. For the second part of the problem, take it for granted that $\frac{1}{7!\cdot 22} < \frac{1}{100{,}000} < \frac{1}{5!\cdot 16}$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

$$\int_0^1 \sin(x^3)\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!\,(6n+4)}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!\,(6n+4)}$$

$$= \frac{1}{4} - \frac{1}{3!\cdot 10} + \frac{1}{5!\cdot 16} - \frac{1}{7!\cdot 22} + \cdots$$

$$\left|\text{value} - \left(\frac{1}{4} - \frac{1}{3!\cdot 10} + \frac{1}{5!\cdot 16}\right)\right| \le \frac{1}{7!\cdot 22} < \frac{1}{100000} \quad \text{but} \quad \frac{1}{5!\cdot 16} \not< \frac{1}{100{,}000}$$

So $\frac{1}{4} - \frac{1}{3!\cdot 10} + \frac{1}{5!\cdot 16}$ is desired partial sum

7. Find the general solution of each of the following nonhomogeneous differential equations.

(a) (10 points) $y'' + 10y' + 25y = 5x - e^{2x}$

$r^2 + 10r + 25 = 0 \Rightarrow r = 5 \qquad y_h = c_1 e^{5x} + c_2 x e^{5x}$

$\underline{G(x) = 5x} \qquad y_{p_1} = A + Bx$

$0 + 10(B) + 25(A + Bx) = 5x$

$\Rightarrow 10B + 25A = 0 \qquad 25B = 5$

$2 + 25A = 0 \qquad B = 1/5$

$A = -\frac{2}{25}$

$\underline{G(x) = -e^{2x}} \qquad y_{p_2} = Ae^{2x} \qquad 4Ae^{2x} + 10(2Ae^{2x}) + 25(Ae^{2x}) = -e^{2x}$

$4A + 20A + 25A = -1$

$A = -1/49$

$y = c_1 e^{5x} + c_2 x e^{5x} - \frac{2}{25} + \frac{1}{5}x - \frac{1}{49}e^{2x}$

(b) (10 points) $y'' + 3y = 7\sin x + \cos x$

$r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}\, i$

$y_h = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$

$y_p = A\sin x + B\cos x$

$-A\sin x - B\cos x + 3(A\sin x + B\cos x) = 7\sin x + \cos x$

$2A = 7 \qquad 2B = 1$

$A = 7/2 \qquad B = 1/2$

$y = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x) + \frac{7}{2}\sin x + \frac{1}{2}\cos x$

8. (10 points) Find the 5th order Taylor polynomial centered at 0 of the solution of the following differential equation with given initial conditions:

$$y'' + x^2y' - 4y = 0,\ y(0) = 2,\ y'(0) = 5$$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad c_0 = y(0) = 2$$

$$c_1 = y'(0) = 5$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots)$$

$$+ (c_1x^2 + 2c_2x^3 + \cdots)$$

$$-(4c_0 + 4c_1x + 4c_2x^2 + 4c_3x^3 + \cdots) = 0$$

$$2c_2 - 4c_0 = 0 \Rightarrow c_2 = 2c_0 = 4$$

$$6c_3 - 4c_1 = 0 \Rightarrow c_3 = \frac{2}{3}c_1 = \frac{10}{3}$$

$$12c_4 + c_1 - 4c_2 = 0 \Rightarrow c_4 = \frac{1}{3}c_2 - \frac{1}{12}c_1$$

$$= \frac{10}{9} - \frac{5}{12}$$

$$20c_5 + 2c_2 - 4c_3 = 0 \Rightarrow c_5 = \frac{1}{5}c_3 - \frac{1}{10}c_2$$

$$= \frac{2}{3} - \frac{2}{5}$$

5th order poly. $= 2 + 5x + 4x^2 + \frac{10}{3}x^3 + \left(\frac{10}{9} - \frac{5}{12}\right)x^4 + \left(\frac{2}{3} - \frac{2}{5}\right)x^5$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.