Northwestern University

Math 226 Midterm 1
Fall Quarter 2019
October 22, 2019

Name: SOLUTIONS

Initials:

netID:

Instructions

- This examination consists of 7 pages, not including this cover page. Verify that your copy of this examination contains all 7 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- Enter your initials in the indicated box on each page, and enter your Name and netID on the indicated boxes on the cover sheet.
- This examination consists of 5 questions for a total of 100 points.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly. Cross out any work that you do not wish to have scored.
- Show all of your work and thoroughly explain your reasoning. Unsupported answers may not earn credit.

Scoring

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Provide justification as to why each of the following sequences converge.

(a) (10 points) $a_n = \sin[(1+n)^{1/n}]$

1st $\lim_{n\to\infty} \ln(1+n)^{1/n} = \lim_{n\to\infty} \frac{\ln(1+n)}{n} = \lim_{n\to\infty} \frac{1/(1+n)}{1} = 0$

L'Hopital

2nd $\lim_{n\to\infty} (1+n)^{1/n} = \lim_{n\to\infty} e^{\ln(1+n)^{1/n}} = e^{\lim_{n\to\infty} \ln(1+n)^{1/n}}$

e^x continuous

$= e^0 = 1$

3rd $\lim_{n\to\infty} \sin[(1+n)^{1/n}] = \sin\left(\lim_{n\to\infty} (1+n)^{1/n}\right) = \sin 1$

$\sin x$ continuous

(b) (10 points) $b_n = \frac{5\sin^2(n^2)+3\cos(n^3)}{n+1}$

$$\frac{5\cdot 0+3(-1)}{n+1} \le \frac{5\sin^2(n^2)+3\cos(n^3)}{n+1} \le \frac{5\cdot 1+3\cdot 1}{n+1} = \frac{8}{n+1}$$

$$\| \qquad -\frac{3}{n+1}$$

Since $\lim_{n\to\infty} \frac{8}{n+1} = 0$ and $\lim_{n\to\infty} \frac{-3}{n+1} = 0$,

$\lim_{n\to\infty} b_n = 0$ by the Sandwich theorem

2. (20 points) Justify the fact that the following series converges, and determine the value to which it converges. Your answer for the value should be left in unsimplified form.

$$\sum_{n=3}^{\infty} \frac{2 \cdot 3^{n+2}}{5 \cdot 7^{n-1}}$$

Rewrite as

$$\sum_{n=3}^{\infty} \frac{2 \cdot 3^n \cdot 3^2}{5 \cdot 7^n \cdot 7^{-1}} = \frac{2 \cdot 3^2}{5 \cdot 7^{-1}} \sum_{n=3}^{\infty} \left(\frac{3}{7}\right)^n$$

Since $\left|\frac{3}{7}\right| < 1$, this geometric series converges

$$\sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n = \frac{1}{1-\frac{3}{7}}, \text{ so}$$

$$\sum_{n=3}^{\infty} \left(\frac{3}{7}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n - \left(\frac{3}{7}\right)^0 - \left(\frac{3}{7}\right)^1 - \left(\frac{3}{7}\right)^2$$

$$= \frac{1}{1-\frac{3}{7}} - 1 - \frac{3}{7} - \frac{3^2}{7^2}.$$

Thus

$$\sum_{n=3}^{\infty} \frac{2 \cdot 3^{n+2}}{5 \cdot 7^{n-1}} = \frac{2 \cdot 3^2}{5 \cdot 7^{-1}} \sum_{n=3}^{\infty} \left(\frac{3}{7}\right)^n$$

$$= \frac{2 \cdot 3^2}{5 \cdot 7^{-1}} \left(\frac{1}{1-\frac{3}{7}} - 1 - \frac{3}{7} - \frac{3^2}{7^2} \right)$$

3. (20 points) Use the integral test to verify that the following series converges. Be sure to explain why the integral test is applicable. (Note: integration by parts is NOT needed.)

$$\sum_{n=4}^{\infty} ne^{-n^2}$$

Set $f(x) = xe^{-x^2}$. This is positive on $[4, \infty)$

and $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0$,

so $f(x)$ is decreasing.

Also, $f(x)$ is continuous, so integral test applies.

$$\int_4^\infty xe^{-x^2}dx = \lim_{b\to\infty} \int_4^b xe^{-x^2}dx$$

$$= \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_4^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left[e^{-b^2} - e^{-16}\right]$$

$$= \frac{1}{2}e^{-16}.$$

Since $\int_4^\infty xe^{-x^2}dx$ converges, $\sum_{n=4}^{\infty} ne^{-n^2}$

converges as well by the integral test.

4. Determine whether each of the following series converge or diverge. Justify your answer and be clear about which convergence test you are using.

(a) (10 points) $\sum_{n=1}^{\infty} a_n$, where a_n is the sequence from Problem 1 part (a).

$$a_n = \sin\left[(1+n)^{1/n}\right]$$

Since $\lim_{n\to\infty} a_n = \sin 1 \neq 0$,

Series $\sum a_n$ diverges by n-term test

(b) (10 points) $\sum_{n=2}^{\infty} \frac{n^3+2n}{n^9+1}$

Set $a_n = \frac{n^3+2n}{n^9+1}$, $b_n = \frac{1}{n^6}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n^3+2n}{n^9+1}\right)\frac{n^6}{1}$$

$$= \lim_{n\to\infty} \frac{n^9+2n^7}{n^9+1} = \lim_{n\to\infty} \frac{1+2/n^2}{1+1/n^2} = 1$$

Since $\sum b_n$ converges by p-series test,

$\sum a_n$ converges by limit comparison test

5. (20 points) This problem deals with the following alternating series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{e^{2n}}$$

Verify that this series converges, and find a restriction on how large the difference between the 3-rd partial sum of this series and the actual value to which it converges can be. (This restriction can be left in an unsimplified form.)

Set $b_n = \frac{n+2}{e^{2n}}$. This is positive and

$$\lim_{n\to\infty} \frac{n+2}{e^{2n}} \underset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{1}{2e^{2n}} = 0.$$

For $f(x) = \frac{x+2}{e^{2x}}$, $f'(x) = \frac{e^{2x} - (x+2)2e^{2x}}{e^{4x}}$

$$= \frac{(-3-2x)e^{2x}}{e^{4x}} < 0$$

So $f(n) = b_n$ decreases.

Thus $\sum (-1)^n b_n$ converges by alternating series test

$$\left| \text{value} - \left(3^{rd} \text{ partial sum}\right) \right| \le b_4 = \frac{6}{e^8}$$

So difference between actual value and partial sum approximation is no more than $6/e^8$.

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