Northwestern University

Math 226 Midterm 2
Fall Quarter 2019
November 18, 2019

Name: SOLUTIONS

Initials:

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Instructions

- This examination consists of 7 pages, not including this cover page. Verify that your copy of this examination contains all 7 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- Enter your initials in the indicated box on each page, and enter your Name and netID on the indicated boxes on the cover sheet.
- This examination consists of 5 questions for a total of 100 points.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly. Cross out any work that you do not wish to have scored.
- Show all of your work and thoroughly explain your reasoning. Unsupported answers may not earn credit.

Scoring

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Find the interval of convergence of the following power series.

$$\sum_{n=0}^{\infty} \underbrace{\frac{3^n}{2^{n-1}(n+5)}(x-1)^n}_{a_n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{2^n(n+6)}|x-1|^{n+1}}{\frac{3^n}{2^{n-1}(n+5)}|x-1|^n}$$

$$= \lim_{n\to\infty} \frac{3}{2}\,\frac{n+5}{n+6}\,|x-1|$$

$$= \frac{3}{2}|x-1| \lim_{n\to\infty} \frac{1+5/n}{1+6/n} = \frac{3}{2}|x-1|$$

converges for $\frac{3}{2}|x-1|<1$ by ratio test,

so for $|x-1|<2/3$.

interval is at least $\left(1-\frac{2}{3}, 1+\frac{2}{3}\right) = \left(\frac{1}{3}, \frac{5}{3}\right)$

$\underline{x=\frac{1}{3}}$ $\quad \sum_{n=0}^{\infty} \frac{3^n}{2^{n-1}(n+5)}\left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+5)}$ converges

by alternating series test

$\underline{x=\frac{5}{3}}$ $\quad \sum_{n=0}^{\infty} \frac{3^n}{2^{n-1}(n+5)}\left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2(n+5)}$ diverges by limit comparison with $\sum \frac{1}{n}$.

Interval is $\left[\frac{1}{3}, \frac{5}{3}\right)$

2. Find power series which represent each of the functions below around the given centers.

(a) (10 points) $f(x) = x^2 \sin(x^5)$ centered at 0

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(x^5) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^5)^{2n+1}}{(2n+1)!}$$

$$x^2 \sin(x^5) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+7}}{(2n+1)!}$$

(b) (10 points) $g(x) = \int_0^x \frac{1}{2-(t-3)^4}\, dt$ centered at 3 (Hint: factor 2 out of the denominator of the function being integrated.)

$$\frac{1}{2-(t-3)^4} = \frac{1}{2\left(1 - \frac{(t-3)^4}{2}\right)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{(t-3)^4}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(t-3)^{4n}}{2^{n+1}}$$

$$\int_0^x \frac{1}{2-(t-3)^4}\, dt = \sum_{n=0}^{\infty} \frac{(t-3)^{4n+1}}{2^{n+1}(4n+1)} \Bigg|_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(x-3)^{4n+1}}{2^{n+1}(4n+1)} - \sum_{n=0}^{\infty} \frac{(-3)^{4n+1}}{2^{n+1}(4n+1)}$$

3. (a) (10 points) Find the third-order Taylor polynomial centered at 2 of the function $f(x) = \sqrt{1+x}$.

$f'(x) = \frac{1}{2(1+x)^{1/2}}$ $f''(x) = -\frac{1}{4(1+x)^{3/2}}$

$f'''(x) = \frac{3}{8}\frac{1}{(1+x)^{5/2}}$

$f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$= \sqrt{3} + \frac{1}{2\sqrt{3}}(x-2) - \frac{1}{8\sqrt{3}^3}(x-2)^2 + \frac{3}{48\sqrt{3}^5}(x-2)^3$

(b) (10 points) Find the Taylor series centered at 1 of the function $g(x) = x + x^2 - 2x^3$.

$g'(x) = 1 + 2x - 6x^2$ $g''(x) = 2 - 12x$

$g'''(x) = -12$ $g^{(n)}(x) = 0$ for $n \geq 4$.

$\sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(x-1)^n$

$= g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3$

$= -3(x-1) - \frac{10}{2}(x-1)^2 - \frac{12}{3!}(x-1)^3$

4. Consider the function $f(x) = e^{3x}$

(a) (10 points) Determine the maximal error which arises when approximating $f(x)$ using the polynomial $1 + 3x + \frac{9}{2}x^2$ for x in the interval $(-2, 2)$.

$$f(0) + f'(0)x + f''(0)x^2 = 1 + 3x + \frac{9}{2}x^2$$

So $|\text{error}| = \left| \frac{f'''(c)}{3!} \right| |x|^3$ for some c between 0 and x

x in $(-2,2) \Rightarrow c$ in $(-2,2)$

So $|\text{error}| = \frac{27e^{3c}}{6}|x|^3 \leq \underline{\underline{\frac{27e^6}{6} \cdot 2^3}}$

(b) (10 points) Find the values of x for which the error in approximating $f(x)$ using the polynomial $1 + 3x$ is at most $\frac{1}{100}$. Your answer should be an interval characterizing such x, but the endpoints of this interval do not have be written in simplified form.

$$f(0) + f'(0)x = 1 + 3x$$

$|\text{error}| = \left| \frac{f''(c)}{2} x^2 \right|$ for some c between 0 and x

$= \frac{9e^{3c}}{2}|x^2| \leq \frac{9e^3}{2}|x^2|$ assuming $c < 1$

<u>Want</u> $\frac{9e^3}{2}|x|^2 \leq \frac{1}{100} \Rightarrow |x| \leq \sqrt{\frac{2}{900e^3}}$

So x in $\left[-\sqrt{\frac{2}{900e^3}}, \sqrt{\frac{2}{900e^3}}\right]$.

less than 1, so $c < 1$ valid.

5. Find all solutions of the following second-order differential equations.

(a) (10 points) $y'' - 3y' - 28y = 0$

$r^2 - 3r - 28 = 0$

$(r-7)(r+4) = 0 \quad r = -4, 7$

$y = c_1 e^{-4x} + c_2 e^{7x}$

(b) (10 points) $y'' - 4y' + 5y = 0$

$r^2 - 4r + 5 = 0$

$r = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$e^{(2+i)x} = e^{2x} e^{ix} = e^{2x}(\cos x + i \sin x)$

$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$

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