Northwestern University

MATH 230-1 Final Exam
Fall Quarter 2023
December 5, 2023

Last name: SOLUTIONS Email address: ____________

First name: ____________ NetID: ____________

Instructions

- Mark your section.

Section	Time	Instructor	
31	9:00	Lee	
41	10:00	Lee	
51	11:00	Wunsch	
61	12:00	Cañez	
71	1:00	Coles	
81	2:00	Coles	

- This examination consists of 13 pages, not including this cover page. Verify that your copy of this examination contains all 13 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 100 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones. Use only material covered in this course (i.e., in the textbook or lecture) and not any formulas you may know from elsewhere that we did not cover.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has four parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

(a) (5 points) The triangle with vertices $(0,0,0)$, $(-2,2,1)$, and $(-5,1,2)$ has area larger than 4.

Sides are $\langle -2,2,1\rangle$ and $\langle -5,1,2\rangle$

$$\langle -2,2,1\rangle \times \langle -5,1,2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ -5 & 1 & 2 \end{vmatrix} = \langle 3,-1,8\rangle$$

area of triangle $= \frac{1}{2}$ (area of parallelogram) $= \frac{1}{2}\,|\text{cross product}|$

$$= \frac{1}{2}\sqrt{9+1+64} = \frac{1}{2}\sqrt{74} > \frac{1}{2}\sqrt{64} = 4$$

$\boxed{\text{TRUE}}$

(b) (5 points) The cross-sections of the paraboloid $z = 2x^2 + 4y^2$ at fixed $x = k$ are ellipses.

$z = 2k^2 + 4y^2$ is a parabola

$\boxed{\text{FALSE}}$

(c) (5 points) The limit $\lim_{(x,y)\to(0,3)} \dfrac{x^2 - x + 2y^2}{(x^2+y^2)^4}$ exists.

function is continuous at $(0,3)$

so $\lim_{(x,y)\to(0,3)} \dfrac{x^2-x+2y^2}{(x^2+y^2)^4} = \dfrac{0^2-0+2\cdot 3^2}{(0^2+3^2)^4} = \dfrac{18}{9^4}$

$\boxed{\text{TRUE}}$

(d) (5 points) There is a point $(a, b, f(a, b))$ on the graph of $f(x, y) = 2x^3y$ at which the tangent plane is parallel to the plane $z = 12x + 2y - 100$.

tangent plane is

$z = 2a^3b + 6a^2b(x-a) + 2a^3(y-b)$

For $a=1$ and $b=2$ get

$z = 4 + 12(x-1) + 2(y-2)$, which

is parallel to $z = 12x + 2y - 100$

$\boxed{\text{TRUE}}$

2. (10 points) Consider the intersecting lines with parametric equations

$$\begin{cases} x = -8 + t \\ y = t \\ z = 14 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = -3t \\ y = 3 + 2t \\ z = 1 - 2t. \end{cases}$$

Find parametric equations for the line which is perpendicular to both of these lines and passes through their point of intersection. If you are having trouble finding the point of intersection, use (x_0, y_0, z_0) in place of this point for partial credit.

$-8 + t_1 = -3t_2 \longrightarrow -8 + (3 + 2t_2) = -3t_2$

$t_1 = 3 + 2t_2 \rightarrow$ plug into 1st

$14 - 3t_1 = 1 - 2t_2$

$-5 = -5t_2$

so $t_2 = 1$

$t_1 = 3 + 2(1) = 5$

check z-coordinates agree :

$14 - 3(5) = -1 = 1 - 2(1)$

intersection is $(-3, 5, -1)$

perpendicular direction vector

$$= \begin{pmatrix} \text{direction} \\ \text{line 1} \end{pmatrix} \times \begin{pmatrix} \text{direction} \\ \text{line 2} \end{pmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -3 & 2 & -2 \end{vmatrix} = \langle 4, 11, 5 \rangle$$

line is $x = -3 + 4t$, $y = 5 + 11t$, $z = -1 + 5t$

3. Consider the surfaces with equations $y = 3x$ and $x^2 + z^2 = 4$.

(a) (5 points) Find a set of parametric equations, with correct parameter bounds, for the portion of the curve where these surfaces intersect that lies in the region where $z \geq 0$.

1^{st} approach: $x = t$, $y = 3t$, $z = \sqrt{4-t^2}$, $-2 \leq t \leq 2$

positive square root since $z \geq 0$

the bounds on t come from solving $z = \sqrt{4-t^2} \geq 0$.

2^{nd} approach: $x = 2\cos t$, $y = 6\cos t$, $z = 2\sin t$

$0 \leq t \leq \pi$

(b) (5 points) Setup an integral which gives the arclength of this portion of the curve of intersection. Simplify your integral expression as much as you can without actually evaluating it.

1^{st} approach: $\vec{r}(t) = \langle t, 3t, \sqrt{4-t^2} \rangle$

$\vec{r}\,'(t) = \langle 1, 3, \frac{-2t}{2\sqrt{4-t^2}} \rangle$

arclength $= \int_{-2}^{2} |\vec{r}\,'(t)|\,dt = \int_{-2}^{2} \sqrt{1+9+\frac{t^2}{4-t^2}}\,dt = \int_{-2}^{2} \sqrt{10+\frac{t^2}{4-t^2}}\,dt$

2^{nd} approach: $\vec{r}(t) = \langle 2\cos t, 6\cos t, 2\sin t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, -6\sin t, 2\cos t \rangle$

arclength $= \int_0^{\pi} \sqrt{4\sin^2 t + 36\sin^2 t + 4\cos^2 t}\,dt = \int_0^{\pi} \sqrt{4+36\sin^2 t}\,dt$

4. Consider the function $f(x, y) = y^2 \cos(xy)$.

(a) (5 points) Find a direction in which the rate of change of $f(x, y)$ at $(\frac{\pi}{2}, 1)$ is positive. Justify your answer.

$\nabla f = \langle -y^3 \sin(xy), 2y\cos(xy) - xy^2 \sin(xy) \rangle$

$\nabla f(\frac{\pi}{2}, 1) = \langle -1, -\frac{\pi}{2} \rangle$

$D_{\vec{u}} f(\frac{\pi}{2}, 1) = \langle -1, -\frac{\pi}{2} \rangle \cdot \vec{u} > 0$

for $\vec{u} = \langle -1, 0 \rangle$ for example (Get value of 1)

$\vec{u} = \langle 0, -1 \rangle$ is another example (Get $\frac{\pi}{2}$)

(b) (5 points) Compute f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

$f_x = -y^3 \sin(xy)$

$f_y = 2y\cos(xy) - xy^2 \sin(xy)$

$f_{xx} = -y^4 \cos(xy)$

$f_{yy} = 2\cos(xy) - 2xy\sin(xy) - 2xy\sin(xy) - x^2y^2\cos(xy)$

$f_{xy} = -3y^2 \sin(xy) - xy^3 \cos(xy) = f_{yx}$

5. Consider the surfaces with equations $3z^2 = 1 + x^2 + y^2$ and $2xy + z^2 = 3$.

(a) (5 points) Find normal vectors to each surface at an arbitrary point (x, y, z).

$$\underbrace{3z^2 - x^2 - y^2}_{f(x,y,z)} = 1 \qquad \underbrace{2xy + z^2}_{g(x,y,z)} = 3$$

normal $= \nabla f = \langle -2x, -2y, 6z \rangle$

normal $= \nabla g = \langle 2y, 2x, 2z \rangle$

(b) (5 points) The point $(1, 1, 1)$ lies on both surfaces. Find the cosine of the angle between the normal vectors to each surface at $(1, 1, 1)$. Your answer should be expressed using square roots but can otherwise be left unsimplified.

normal to 1^{st} $= \nabla f(1,1,1) = \langle -2, -2, 6 \rangle = \vec{u}$

normal to 2^{nd} $= \nabla g(1,1,1) = \langle 2, 2, 2 \rangle = \vec{v}$

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-4-4+12}{\sqrt{4+4+36}\sqrt{4+4+4}}$$

$$= \frac{4}{\sqrt{44}\sqrt{12}}$$

6. Consider the function $f(x, y) = \frac{e^{2x}}{y}$.

(a) (5 points) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at $(0, 1, 1)$.

$$f_x = \frac{2e^{2x}}{y} \qquad f_y = -\frac{e^{2x}}{y^2}$$

$$z = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)$$

$$z = 1 + 2x - (y-1)$$

(b) (5 points) Justify the fact that using the tangent plane found in (a) to approximate the value of $\frac{e^{2(0.3)}}{0.9}$ results in an error no larger than $\frac{1}{2}\frac{4e}{(0.9)^3}(0.4)^2$. (Using the tangent plane to approximate the value is the same as using what the book calls the linearization to approximate the value.)

For $0 \le x \le 0.3$, $0.9 \le y \le 1$

$$|f_{xx}| = \left|\frac{4e^{2x}}{y}\right| \le \frac{4e^{0.6}}{0.9} \le \frac{4e}{(0.9)^3}$$

$$|f_{xy}| = \left|-\frac{2e^{2x}}{y^2}\right| \le \frac{2e^{0.4}}{(0.9)^2} \le \frac{2e}{(0.9)^3} < \frac{4e}{(0.9)^3}$$

$$|f_{yy}| = \left|\frac{2e^{2x}}{y^3}\right| \le \frac{2e^{0.4}}{(0.9)^3} \le \frac{4e}{(0.9)^3}$$

So error $\le \frac{1}{2}\left(\frac{4e}{(0.9)^3}\right)(\underbrace{|\Delta x|}_{0.3} + \underbrace{|\Delta y|}_{0.1})^2 = \frac{1}{2}\left(\frac{4e}{(0.9)^3}\right)(0.4)^2$.

7. Consider the function $f(x,y) = 6xy - x^3 - 3y^2$.

(a) (10 points) Find all critical points of $f(x,y)$ and determine whether each is a local minimum, a local maximum, or a saddle point.

$f_x = 6y - 3x^2 = 0 \xrightarrow{\text{plug in } x=y} 6y - 3y^2 = 0$

$f_y = 6x - 6y = 0 \rightarrow x = y$

$y=0$ or $y=2$

$x=0$ $\quad$ $x=2$

critical points $(0,0), (2,2)$

$f_{xx} = -6x \quad f_{xy} = 6 \quad f_{yy} = -6$

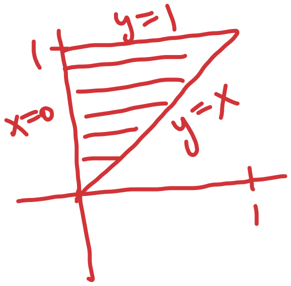
@ $(0,0)$: $f_{xx} f_{yy} - (f_{xy})^2 = 0 - 36 < 0$

saddle point

@ $(2,2)$: $f_{xx} f_{yy} - (f_{xy})^2 = (-12)(-6) - 36 > 0$ and $f_{xx} = -12 < 0$

local max

(b) (10 points) Find the points at which $f(x,y)$ has an absolute maximum or an absolute minimum on the triangle with vertices $(0,0), (0,1)$, and $(1,1)$.



critical point $(2,2)$ is not in region, so ignore

On $x=0$: $f(0,y) = -3y^2$, $f_y = -6y = 0$

$\rightarrow y=0, x=0$

On $y=1$: $f(x,1) = 6x - x^3 - 3$

$f_x = 6 - 3x^2 = 0 \rightarrow x = \pm\sqrt{2}$ not in region

On $y=x$: $f(x,x) = 3x^2 - x^3$

$f_x = 6x - 3x^2 = 0 \rightarrow x=0, y=0$

$x=2$ outside

Test points

$f(0,0) = 0 \quad f(1,1) = 2$

$f(0,1) = -3$

max @ $(1,1)$

min @ $(0,1)$

8. (10 points) Using the method of Lagrange multipliers, find the points on the double cone $z^2 = x^2 + y^2$ that are closest to the point $(1, 2, 0)$. You do not have to justify why the points you find are closest instead of farthest. You can take it for granted that $(0, 0, 0)$ is not the point you want.

minimize $f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$

(square of distance to $(1,2,0)$)

Constraint $g(x,y,z) = z^2 - x^2 - y^2 = 0$

$\nabla f = \lambda \nabla g \longrightarrow \langle 2(x-1), 2(y-2), 2z \rangle = \lambda \langle -2x, -2y, 2z \rangle$

Solve $2(x-1) = -\lambda 2x$

$2(y-2) = -\lambda 2y$

$2z = \lambda 2z \longrightarrow z = 0$ or $\lambda = 1$

$z^2 = x^2 + y^2$

$z = 0 \downarrow$ constraint gives $x=0, y=0$

$(0,0,0)$ not point we want

$\underline{\lambda = 1}$ $2x - 2 = -2x$

$\rightarrow x = \frac{1}{2}$

$2y - 4 = -2y \longrightarrow y = 1$

$z^2 = \left(\frac{1}{2}\right)^2 + 1 = \frac{5}{4}$

So $\left(\frac{1}{2}, 1, \pm\sqrt{5/4}\right)$ are closest points

THERE IS NO EXAMINATION MATERIAL ON THIS PAGE.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

THERE IS NO EXAMINATION MATERIAL ON THIS PAGE.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

THERE IS NO EXAMINATION MATERIAL ON THIS PAGE.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

THERE IS NO EXAMINATION MATERIAL ON THIS PAGE.

DO NOT WRITE ON THIS PAGE.