Northwestern University

MATH 230-1 Midterm 2
Fall Quarter 2023
November 14, 2023

Last name: SOLUTIONS Email address: __________

First name: __________ NetID: __________

Instructions

- Mark your section.

Section	Time	Instructor	
31	9:00	Lee	
41	10:00	Lee	
51	11:00	Wunsch	
61	12:00	Cañez	
71	1:00	Coles	
81	2:00	Coles	

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones. Use only material covered in this course (i.e., in the textbook or lecture) and not any formulas you may know from elsewhere that we did not cover.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has four parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

   (a) (5 points) The arclength of the curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 4\pi$ is larger than 4π.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\text{arclength} = \int_0^{4\pi} |\vec{r}\,'(t)|\,dt = \int_0^{4\pi} \sqrt{2}\,dt = 4\sqrt{2}\pi > 4\pi$$

$\boxed{\text{TRUE}}$

   (b) (5 points) There is a level curve of $f(x, y) = \sin(x^2 + y^2)$ that contains a circle.

level curve at $z = 0$

$$0 = \sin(x^2+y^2)$$

so $x^2+y^2 =$ integer multiple of π

contains for example $x^2+y^2 = \pi$,

which is a circle. $\boxed{\text{TRUE}}$

(c) (5 points) The limit of $f(x, y) = \frac{3xy}{2x^2+5y^2}$ as (x, y) approaches $(0, 0)$ exists.

Along $y=0$: $\lim_{(x,0)\to(0,0)} \frac{0}{2x^2+0} = 0$

Along $y=x$: $\lim_{(x,x)\to(0,0)} \frac{3x^2}{2x^2+5x^2} = \frac{3}{7}$

limit does not exist since are different

$\boxed{\text{FALSE}}$

(d) (5 points) There is a direction in which the directional derivative of $f(x, y) = xe^{xy}$ at $(1, 2)$ is 0. (Either give an example of such a direction vector or justify that none exists.)

$\boxed{\text{TRUE}}$

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$

$\nabla f = \langle e^{xy} + xye^{xy}, x^2e^{xy} \rangle$

$\nabla f(1,2) = \langle 3e^2, e^2 \rangle$

so for $\vec{u}$ in direction of $\langle -1, 3\rangle$ or $\langle 1, -3\rangle$

get $D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = 0$

Check $\vec{u} = \langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$, $\langle 3e^2, e^2\rangle \cdot \langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = \frac{-3e^2}{\sqrt{10}} + \frac{3e^2}{\sqrt{10}}$

$= 0$

2. Suppose a rocket moves through space with acceleration at time t given by

$$\mathbf{a}(t) = 2\,\mathbf{i} - 6t\,\mathbf{j} + e^t\,\mathbf{k}.$$

The rocket starts at the point $(1, 2, 3)$ with initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(a) (10 points) Find the position vector of the rocket at an arbitrary time t

$\vec{v}(t) = \int \vec{a}(t)\,dt = (2t + c_1)\vec{i} + (-3t^2 + c_2)\vec{j} + (e^t + c_3)\vec{k}$

$\vec{v}(0) = \vec{i} + \vec{j} - \vec{k}$ requires $c_1 = 1$, $c_2 = 1$, $e^0 + c_3 = -1$ so $c_3 = -2$

so $\vec{v}(t) = (2t+1)\vec{i} + (-3t^2+1)\vec{j} + (e^t - 2)\vec{k}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = (t^2 + t + d_1)\vec{i} + (-t^3 + t + d_2)\vec{j} + (e^t - 2t + d_3)\vec{k}$

$\vec{r}(0) = \vec{i} + 2\vec{j} + 3\vec{k}$ requires $d_1 = 1$, $d_2 = 2$, $e^0 + d_3 = 3$ so $d_3 = 2$

position $\vec{r}(t) = (t^2 + t + 1)\vec{i} + (-t^3 + t + 2)\vec{j} + (e^t - 2t + 2)\vec{k}$

(b) (10 points) Suppose that at time $t = 10$ the rocket engine turns off, so that the rocket begins to follow the path of the tangent line (with no acceleration and velocity equal to the tangent vector) at the point with position vector $\mathbf{r}(10)$. Find parametric equations for the path along the tangent line the rocket will follow.

$\vec{r}(10) = 111\vec{i} - 988\vec{j} + (e^{10} - 18)\vec{k}$

$\vec{r}\,'(t) = v(t) = (2t+1)\vec{i} + (-3t^2+1)\vec{j} + (e^t - 2)\vec{k}$

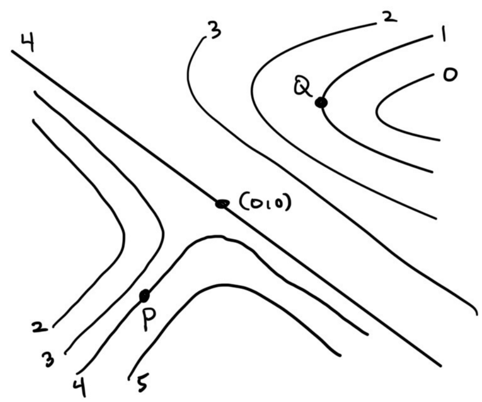
$\vec{r}\,'(10) = 21\vec{i} - 299\vec{j} + (e^{10} - 2)\vec{k}$

line through $(111, -988, e^{10} - 18)$ parallel to $\langle 21, -299, e^{10} - 2\rangle$:

$x = 111 + 21t$, $y = -988 - 299t$,

$z = e^{10} - 18 + (e^{10} - 2)t$

3. (This problem has two parts and continues on the next page.) Suppose $z = f(x, y)$ is a continuous function with nonnegative values and the following level curves:



The labels on the level curves are the values of z at which they occur. Assume that other level curves occur at values of z strictly between those that are drawn. The x-axis (not drawn) passes through $(0, 0)$ horizontally, and the y-axis (not drawn) passes through $(0, 0)$ vertically.

(a) (10 points) Assuming that f_x and f_y exist and are continuous at all points, determine whether each of the following is positive, negative, or zero.

$$f_x(P), \quad f_y(P), \quad f_x(Q) - f_y(Q)$$

Justify your answer for $f_x(Q) - f_y(Q)$ only.

$f_x(P) > 0$ $f_y(P) < 0$

$f_x(Q) < 0$ since f decreases in value through Q in direction of increasing x

$f_y(Q) = 0$ since Q is at a local minimum of f when varying only the y-coordinate: $f(Q) = 1$ and $f(x,y) > 1$ for points above and below Q

so $\underbrace{f_x(Q)}_{\text{neg.}} - \underbrace{f_y(Q)}_{\text{zero}} < 0$

(b) (10 points) Determine, with appropriate justification, the value of the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2f(x,y)}{(x^2+y^2)^{3/2}}$$

Convert to polar:

$$\frac{r^2\cos^2\theta\, r^2\sin^2\theta\, f(r\cos\theta, r\sin\theta)}{(r^2)^{3/2}}$$

$$= r\cos^2\theta\sin^2\theta\, f(r\cos\theta, r\sin\theta)$$

We have

$$-r f(r\cos\theta, r\sin\theta) \le r\cos^2\theta\sin^2\theta\, f(r\cos\theta, r\sin\theta) \le r^2 f(r\cos\theta, r\sin\theta)$$

Since f is continuous,

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = f(0,0) = 4$$

So $\displaystyle\lim_{r\to 0} \pm r f(r\cos\theta, r\sin\theta) = 0\cdot 4 = 0.$

Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2f(x,y)}{(x^2+y^2)^{3/2}} = 0$

by sandwich theorem

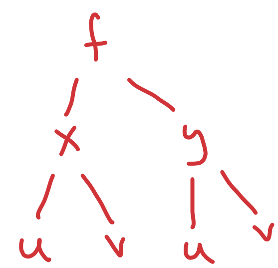
4. Suppose $f(x, y)$ is a function with continuous first-order partial derivatives satisfying

$$f_x(-1,4) = 4, \qquad f_x(2,1) = 3, \qquad f_y(-1,4) = -1, \qquad \text{and} \qquad f_y(2,1) = -2.$$

(a) (10 points) If the coordinates (x, y) of a point depend on variables u and v via

$$x = u - 3v \quad \text{and} \quad y = u^2 v,$$

find the value of $\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$ at $(u, v) = (2, 1)$.



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -3\frac{\partial f}{\partial x} + u^2\frac{\partial f}{\partial y}$$

At $(u,v) = (2,1)$, $(x,y) = (-1,4)$, so

$$\frac{\partial f}{\partial u} = 4 + 4(-1) = 0$$

$$\frac{\partial f}{\partial v} = -3(4) + 4(-1) = -16$$

$$\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = -16$$

(b) (10 points) Find parametric equations for the line which is perpendicular to the curve with Cartesian equation $f(x, y) = f(2, 1)$ at $(2, 1)$.

$\nabla f(2,1)$ is orthogonal to curve, so gives direction of perpendicular line

$$\nabla f(2,1) = \langle f_x(2,1), f_y(2,1)\rangle = \langle 3, -2\rangle$$

line through $(2,1)$ parallel to $\langle 3, -2\rangle$: $x = 2 + 3t$, $y = 1 - 2t$

5. Let $f(x,y) = \frac{3e^y}{x^2-y}$.

(a) (10 points) Find the rate at which f is changing at $(-3,1)$ in the direction of $\frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$.

$$f_x = \frac{-3e^y(2x)}{(x^2-y)^2} \qquad f_y = \frac{(x^2-y)3e^y - (3e^y)(-1)}{(x^2-y)^2}$$

$$\nabla f(-3,1) = \left\langle \frac{18e}{64}, \frac{27e}{64} \right\rangle$$

$$D_{\vec{u}} f(-3,1) = \nabla f(-3,1) \cdot \vec{u}$$

$$= \left\langle \frac{18e}{64}, \frac{27e}{64} \right\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \frac{(36+81)e}{64\sqrt{13}}$$

(b) (10 points) Find the direction in which the rate of change of f at $(-3,1)$ is as negative as possible and the rate of change in this direction.

$$D_{\vec{u}} f(-3,1) = \nabla f(-3,1) \cdot \vec{u} = |\nabla f(-3,1)| \cos\theta$$

minimized when $\theta = \pi$, so

direction of $-\nabla f(-3,1) = \left\langle -\frac{18e}{64}, -\frac{27e}{64} \right\rangle$

rate of change in this direction is $-|\nabla f(-3,1)|$

(when $\theta = \pi$, so $\cos\theta = -1$) $\quad = -\sqrt{\left(\frac{18e}{64}\right)^2 + \left(\frac{27e}{64}\right)^2}$

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