Northwestern

Math 290-1 Final Examination
Fall Quarter 2018
December 12, 2018

Last name: Solutions ______ Email address: ______

First name: ______ NetID: ______

Instructions

- Mark your instructor's name.

  ____ Cañez

  ____ Newstead

  ____ Norton

- This examination consists of 15 pages, not including this cover page. Verify that your copy of this examination contains all 15 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 150 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- **Show all of your work and justify all your answers, unless explicitly directed not to. Unsupported answers may not earn credit.**

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **six** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

(a) (5 points) If A is an $n \times n$ matrix and $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$, then there is exactly one matrix B satisfying $A^3 B = A$.

TRUE

Since $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$, then A is invertible. Thus $A^3B = A \Leftrightarrow B = A^{-3}A = A^{-2} = (A^2)^{-1}$

Since inverses are unique, this implies B is unique.

(b) (5 points) If A is a 4×4 matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ and $\det A = 8$, then

$$\det \begin{bmatrix} -2\vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2 + 3\vec{v}_4 \end{bmatrix} = 16.$$

FALSE

$$\det \begin{bmatrix} -2\vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2+3\vec{v}_4 \end{bmatrix} = -2 \det \begin{bmatrix} \vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2+3\vec{v}_4 \end{bmatrix} = 2 \det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_2+3\vec{v}_4 \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_4 \end{bmatrix} = 6 \det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} = (6)(8) = 48$$

(c) (5 points) There exists a real 3×3 matrix A such that $A^2 = -I_3$.

FALSE

$\det A^2 = (\det A)^2 \geq 0$ for any A, since $\det A$ is a real number.

$\det(-I_3) = (-1)^3 = -1 < 0$

Note to reader: Yes, A^2 is diagonalizable (since A^2 is diagonal). This does not imply A is diagonalizab (or diagonal). Also, it is possible for A^2 to have negative entries. Ex: If A is rotation by $\frac{\pi}{2}$ in $\mathbb{R}^2$, then A^2 is rotation by π, ie $A^2 = -I_2$.

(d) (5 points) The eigenvalues of an $n \times n$ matrix A are the same as the eigenvalues of $\text{rref}(A)$.

FALSE

The eigenvalues of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are $2, 2$.

$\text{rref}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has eigenvalues

$1, 1$.

(e) (5 points) Suppose A is the 2×2 matrix of reflection across the line $y = -x$. Then $A^{100} = A^{51}$.

FALSE.

Since A is reflection across a line, $A^2 = I$.

Therefore $A^{100} = (A^2)^{50} = I$

However $A^{51} = (A^{50})A = (A^2)^{25} A = I^{25} A = A \neq I$

(f) (5 points) Let A be a 7×5 matrix whose kernel is spanned by the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\text{rank}(A) = 2$.

TRUE

$$\begin{bmatrix} | & | & | & | \\ \vec{a} & \vec{b} & \vec{c} & \vec{d} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\vec{a}$, $\vec{b}$, and $\vec{c}$ correspond to pivot columns in the matrix, and $\vec{d}$ is in their span, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for ker A. Thus dim(ker A) = 3. By Rank-Nullity Theorem,

$$\text{rank } A = \#\text{columns of } A - \dim(\ker A) = 5 - 3 = 2.$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **six** parts.)

(a) (5 points) Let A be a 2×2 matrix with characteristic polynomial $\lambda^2 - 3\lambda + 2$, and let $\vec{v}$ be an eigenvector of A. Then $A^2\vec{v} = 3A\vec{v} - 2\vec{v}$.

Since $\vec{v}$ is an eigenvector, then $A^2\vec{v} = \lambda^2\vec{v}$, where λ is the eigenvalue for $\vec{v}$.

Also, $3A\vec{v} - 2\vec{v} = 3\lambda\vec{v} - 2\vec{v}$.

Since λ is an eigenvalue of A, we know it satisfies the characteristic equation, ie

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda^2 = 3\lambda - 2$.

Thus $\lambda^2\vec{v} = 3\lambda\vec{v} - 2\vec{v}$.

$\Rightarrow$ ALWAYS

(b) (5 points) If A is an invertible $n \times n$ matrix, then A is diagonalizable.

SOMETIMES.

Case 1: I_2 is invertible and it is diagonalizable

Case 2: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible because its determinant is 1. However, it is not diagonalizable because $almu(1) = 2$ but $E_1 = \ker(A - I) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. So $gemu(1) = 1$.

(c) (5 points) Let λ be a real eigenvalue of an $n \times n$ matrix A. If $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r)$ is a basis of E_λ, then $(A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_r)$ is a basis of E_λ. (E_λ denotes the eigenspace of A corresponding to λ.)

SOMETIMES

Case 1: If $\lambda \neq 0$, then $A\vec{v}_1 = \lambda\vec{v}_1$, $A\vec{v}_2 = \lambda\vec{v}_2$, ..., $A\vec{v}_n = \lambda\vec{v}_n$. Since $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent and span E_λ, then $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_r$ are linearly independent and span E_λ.

Case 2: If $\lambda = 0$, then $A\vec{v}_1 = A\vec{v}_2 = \ldots = A\vec{v}_r = \vec{0}$. Thus $(A\vec{v}_1, \ldots, A\vec{v}_r)$ is not a basis for E_λ, since $A\vec{v}_1 = \vec{0}$, so the set of vectors is linearly dependent.

(d) (5 points) Let $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^3$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(\vec{x}) = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix}$.

Then the dimension of ker T is 2.

SOMETIMES

Case 1: If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $T(\vec{x}) = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix} = 0$ for all $\vec{x}$ in $\mathbb{R}^3$

$\Rightarrow \ker(T) = \mathbb{R}^3$, so $\dim(\ker T) = 3$.

Case 2: If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, then $T(\vec{x}) = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix} = 0$ if and only if $\vec{x}$ is in span$\{\vec{v}_1, \vec{v}_2\}$

$\Rightarrow \ker T = \text{span}\{\vec{v}_1, \vec{v}_2\}$, so $\dim(\ker T) = 2$

(e) (5 points) Suppose A is an invertible $n \times n$ matrix and λ is a real eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

ALWAYS

Since A is invertible, $\lambda \neq 0$. Moreover, since λ is an eigenvalue of A, there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. By definition of the inverse, we know $A^{-1}(\lambda\vec{v}) = \vec{v}$. Dividing both sides by λ, we get $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$. Thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(f) (5 points) Let A be an invertible 3×3 matrix, and let $\mathfrak{B}$ and $\mathfrak{C}$ be bases of $\mathbb{R}^3$ given by

$$\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \qquad \text{and} \qquad \mathfrak{C} = (A\vec{v}_1, A\vec{v}_2, A\vec{v}_3).$$

If $\vec{x}$ is a vector in $\mathbb{R}^3$, then $[A\vec{x}]_{\mathfrak{C}} = [\vec{x}]_{\mathfrak{B}}$.

ALWAYS. Let $\vec{x}$ be any vector in $\mathbb{R}^3$. Since B is a basis for $\mathbb{R}^3$, there exist scalars c_1, c_2, c_3 such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Thus $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Then $A\vec{x} = A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)$

$= c_1A\vec{v}_1 + c_2A\vec{v}_2 + c_3A\vec{v}_3$. Thus $[A\vec{x}]_C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

$\Rightarrow [\vec{x}]_B = [A\vec{x}]_C$.

3. (15 points) Determine the value(s) of k for which the following matrix is diagonalizable. Justify your answer.

$$\begin{bmatrix} 1 & 0 & 3 \\ 6 & k & 3 \\ 4 & 0 & -3 \end{bmatrix}$$

E-values:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 6 & k-\lambda & 3 \\ 4 & 0 & -3-\lambda \end{vmatrix} = (1-\lambda)\left[(k-\lambda)(-3-\lambda)\right] + 3\left[-(k-\lambda)(4)\right]$$

$$= (k-\lambda)\left[(1-\lambda)(-3-\lambda) - 12\right] = (k-\lambda)\left[-3+2\lambda+\lambda^2-12\right]$$

$$= (k-\lambda)(\lambda^2+2\lambda-12) = (k-\lambda)(\lambda-3)(\lambda+5).$$

E-values: $\lambda = k, 3, -5.$

Note: If $k \neq 3, -5$, then A has 3 distinct e-values, so A is diagonalizable.

What if $k=3$?

$$E_3 = \ker(A-3I) \quad [A-3I \,|\, \vec{0}\,] = \left[\begin{array}{ccc|c} -2 & 0 & 3 & 0 \\ 6 & 0 & 3 & 0 \\ 4 & 0 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 0 & 3 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \dim(E_3) = 1 \Rightarrow \text{gemu}(3) = 1 \neq \text{almu}(3) = 2$

$\Rightarrow A$ is not diagonalizable

What if $k=-5$?

$$E_{-5} = \ker(A+5I) \quad [A+5I \,|\, \vec{0}\,] = \left[\begin{array}{ccc|c} 6 & 0 & 3 & 0 \\ 6 & 0 & 3 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \dim(E_{-5}) = 2 \Rightarrow \text{gemu}(5) = 2 = \text{almu}(5) \Rightarrow A$ is diag.

Therefore A is diagonalizable for all values of k except 3.

4. (15 points) Find all real values of a and b such that $\dim(\operatorname{im} A) = \dim(\ker A)$, where A is the 4×4 matrix defined in terms of a and b by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a+b & a+3 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & a-b & b-1 \end{bmatrix}.$$

Justify your answer.

By Rank-Nullity, we must have $\dim(\operatorname{im} A) + \dim(\ker A) = 4$. Thus $\dim(\operatorname{im} A) = 2 = \dim(\ker A)$.

In order for $\dim(\operatorname{im} A)$ to be 2, A must have only 2 linearly independent columns. We can see that the 1st and 3rd columns are pivot columns, so they are linearly independent. Thus we need the 2nd column to be in the span of the 1st, and we need the 4th column to be in the span of the 3rd. This implies $3a+b = a+3$ and $a-b = b-1$.

$\Rightarrow 2a+b = 3$ and $a - 2b = -1$.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & 1 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow a = 1,\ b = 1.$$

5. (15 points) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates a vector by $-\frac{\pi}{4}$, then reflects the result across the line $y = 2x$, and finally orthogonally projects the result onto the line $y = 3x$. (Recall: The projection of $\vec{x}$ onto the line spanned by a nonzero vector $\vec{v}$ is given by $\text{proj}_{\vec{v}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$. The reflection of $\vec{x}$ across the line spanned by a nonzero vector $\vec{v}$ is given by $\text{ref}_{\vec{v}}\vec{x} = 2\text{proj}_{\vec{v}}\vec{x} - \vec{x}$.)

Rotation $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Reflection: $\text{ref}_{\begin{bmatrix}1\\2\end{bmatrix}}\vec{e}_1 = 2\left(\frac{\vec{e}_1 \cdot \begin{bmatrix}1\\2\end{bmatrix}}{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}}\right)\begin{bmatrix}1\\2\end{bmatrix} - \vec{e}_1 = 2\left(\frac{1}{5}\right)\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{bmatrix}$.

$\text{ref}_{\begin{bmatrix}1\\2\end{bmatrix}}\vec{e}_2 = 2\left(\frac{\vec{e}_2 \cdot \begin{bmatrix}1\\2\end{bmatrix}}{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}}\right)\begin{bmatrix}1\\2\end{bmatrix} - \vec{e}_2 = 2\left(\frac{2}{5}\right)\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\\ \frac{3}{5}\end{bmatrix}$

$\Rightarrow$ Reflection $= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

Projection: $\text{proj}_{\begin{bmatrix}1\\3\end{bmatrix}}\vec{e}_1 = \left(\frac{\vec{e}_1 \cdot \begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}\right)\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{10}\\ \frac{3}{10}\end{bmatrix}$

$\text{proj}_{\begin{bmatrix}1\\3\end{bmatrix}}\vec{e}_2 = \left(\frac{\vec{e}_2 \cdot \begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}\right)\begin{bmatrix}1\\3\end{bmatrix} = \frac{3}{10}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{3}{10}\\ \frac{9}{10}\end{bmatrix}$

$\rightarrow$ Projection $= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

$$A = \frac{1}{10}\cdot\frac{1}{5}\cdot\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{\sqrt{2}}{100}\begin{bmatrix} 9 & 13 \\ 27 & 39 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{100}\begin{bmatrix} -4 & 22 \\ -12 & 66 \end{bmatrix} = \frac{\sqrt{2}}{50}\begin{bmatrix} -2 & 11 \\ -6 & 33 \end{bmatrix}$$

6. (15 points) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\4\\6\end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

If Ω is a region in $\mathbb{R}^3$ with volume $\frac{4}{3}\pi$, find the volume of $T(\Omega)$.

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = T\left(-\frac{1}{2}\begin{bmatrix}0\\0\\-2\end{bmatrix}\right) = -\frac{1}{2}T\begin{bmatrix}0\\0\\-2\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\4\\6\end{bmatrix} = \begin{bmatrix}0\\-2\\-3\end{bmatrix}$$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = T\left(\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\begin{bmatrix}1\\0\\1\end{bmatrix} - T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}0\\-2\\-3\end{bmatrix} = \begin{bmatrix}2\\3\\3\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\end{bmatrix} = T\left(\frac{1}{2}\begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \frac{1}{2}T\begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{1}{2}T\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\3\\3\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix}2 & -1 & 0\\3 & -1 & -2\\3 & -1 & -3\end{bmatrix}$$

$$\det A = 2(3-2) - (-1)(-9+6) = 2-3 = -1.$$

Thus volume of $T(\Omega) = \frac{4}{3}\pi \cdot |\det A| = \frac{4}{3}\pi \cdot |-1| = \frac{4}{3}\pi$

7. (15 points) Suppose $A = \begin{bmatrix} 16 & 7 \\ -12 & -3 \end{bmatrix}$. Find a 2×2 matrix C such that $C^2 = A$. (Hint: The eigenvalues of A are 4 and 9. Diagonalize A.)

$E_4 = \ker(A - 4I)$

$$[A - 4I \mid \vec{0}] = \left[\begin{array}{cc|c} 12 & 7 & 0 \\ -12 & -7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 12 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 12x_1 + 7x_2 = 0.$$

$$\Rightarrow E_4 = \operatorname{span}\left\{\begin{bmatrix} -7 \\ 12 \end{bmatrix}\right\}$$

$E_9 = \ker(A - 9I)$

$$[A - 9I \mid \vec{0}] = \left[\begin{array}{cc|c} 7 & 7 & 0 \\ -12 & -12 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = -x_2$$

$$\Rightarrow E_9 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\Rightarrow A = \underbrace{\begin{bmatrix} -7 & 1 \\ 12 & -1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -7 & 1 \\ 12 & -1 \end{bmatrix}^{-1}}_{S^{-1}}$$

Note: $A = SBS^{-1} = \left(S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1}\right)\left(S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1}\right).$

Thus $C = S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1} = \begin{bmatrix} -7 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \frac{1}{-5} \begin{bmatrix} -1 & -1 \\ -12 & -7 \end{bmatrix} = \begin{bmatrix} \frac{22}{5} & \frac{7}{5} \\ \frac{-12}{5} & \frac{3}{5} \end{bmatrix}.$

8. (15 points) Find a 3×3 matrix A with eigenvalues -2 and 4 such that

$$E_{-2} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_4 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(E_λ denotes the eigenspace of A corresponding to λ.)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

Find the inverse

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$