# Northwestern

Math 290-1 Final Examination
Fall Quarter 2018
December 12, 2018

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

**Instructions**

- Mark your instructor's name.
  - _____ Cañez
  - _____ Newstead
  - _____ Norton
- This examination consists of 15 pages, not including this cover page. Verify that your copy of this examination contains all 15 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 150 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- **Show all of your work and justify all your answers, unless explicitly directed not to. Unsupported answers may not earn credit.**

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **six** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) If $A$ is an $n \times n$ matrix and $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$, then there is exactly one matrix $B$ satisfying $A^3B = A$.

   (b) (5 points) If $A$ is a $4 \times 4$ matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ and $\det A = 8$, then

$$\det \begin{bmatrix} -2\vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2 + 3\vec{v}_4 \end{bmatrix} = 16.$$

(c) (5 points) There exists a real $3 \times 3$ matrix $A$ such that $A^2 = -I_3$.

(d) (5 points) The eigenvalues of an $n \times n$ matrix $A$ are the same as the eigenvalues of $\operatorname{rref}(A)$.

(e) (5 points) Suppose $A$ is the $2 \times 2$ matrix of reflection across the line $y = -x$. Then $A^{100} = A^{51}$.

(f) (5 points) Let $A$ be a $7 \times 5$ matrix whose kernel is spanned by the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\operatorname{rank}(A) = 2$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **six** parts.)

   (a) (5 points) Let $A$ be a $2\times 2$ matrix with characteristic polynomial $\lambda^2 - 3\lambda + 2$, and let $\vec{v}$ be an eigenvector of $A$. Then $A^2\vec{v} = 3A\vec{v} - 2\vec{v}$.

   (b) (5 points) If $A$ is an invertible $n \times n$ matrix, then $A$ is diagonalizable.

(c) (5 points) Let $\lambda$ be a real eigenvalue of an $n \times n$ matrix $A$. If $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r)$ is a basis of $E_\lambda$, then $(A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_r)$ is a basis of $E_\lambda$. ($E_\lambda$ denotes the eigenspace of $A$ corresponding to $\lambda$.)

(d) (5 points) Let $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^3$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(\vec{x}) = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix}$.

Then the dimension of $\ker T$ is 2.

(e) (5 points) Suppose $A$ is an invertible $n \times n$ matrix and $\lambda$ is a real eigenvalue of $A$. Then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

(f) (5 points) Let $A$ be an invertible $3 \times 3$ matrix, and let $\mathfrak{B}$ and $\mathfrak{C}$ be bases of $\mathbb{R}^3$ given by

$$\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \qquad \text{and} \qquad \mathfrak{C} = (A\vec{v}_1, A\vec{v}_2, A\vec{v}_3).$$

If $\vec{x}$ is a vector in $\mathbb{R}^3$, then $[A\vec{x}]_{\mathfrak{C}} = [\vec{x}]_{\mathfrak{B}}$.

3. (15 points) Determine the value(s) of $k$ for which the following matrix is diagonalizable. Justify your answer.

$$\begin{bmatrix} 1 & 0 & 3 \\ 6 & k & 3 \\ 4 & 0 & -3 \end{bmatrix}$$

4. (15 points) Find all real values of $a$ and $b$ such that $\dim(\operatorname{im} A) = \dim(\ker A)$, where $A$ is the $4 \times 4$ matrix defined in terms of $a$ and $b$ by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a+b & a+3 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & a-b & b-1 \end{bmatrix}.$$

Justify your answer.

5. (15 points) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates a vector by $-\frac{\pi}{4}$, then reflects the result across the line $y = 2x$, and finally orthogonally projects the result onto the line $y = 3x$. (Recall: The projection of $\vec{x}$ onto the line spanned by a nonzero vector $\vec{v}$ is given by $\text{proj}_{\vec{v}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$. The reflection of $\vec{x}$ across the line spanned by a nonzero vector $\vec{v}$ is given by $\text{ref}_{\vec{v}}\vec{x} = 2\text{proj}_{\vec{v}}\vec{x} - \vec{x}$. )

6. (15 points) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\4\\6\end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

If $\Omega$ is a region in $\mathbb{R}^3$ with volume $\frac{4}{3}\pi$, find the volume of $T(\Omega)$.

7. (15 points) Suppose $A = \begin{bmatrix} 16 & 7 \\ -12 & -3 \end{bmatrix}$. Find a $2 \times 2$ matrix $C$ such that $C^2 = A$. (Hint: The eigenvalues of $A$ are 4 and 9. Diagonalize $A$. )

8. (15 points) Find a $3 \times 3$ matrix $A$ with eigenvalues $-2$ and $4$ such that

$$E_{-2} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_4 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

($E_\lambda$ denotes the eigenspace of $A$ corresponding to $\lambda$.)

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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