Northwestern

Math 290-1 Midterm I
Fall Quarter 2018
October 22, 2018

Last name: Solutions Email address: ____________

First name: ____________ NetID: ____________

Instructions

- Mark your instructor's name.
  - _____ Cañez
  - _____ Newstead
  - _____ Norton
- This examination consists of 10 pages, not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

(a) (5 points) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.

Answer: TRUE

Since T is linear, $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars k. Let $\vec{v}$ be any vector in $\mathbb{R}^n$. Then $T(\vec{0}) = T(0\cdot\vec{v}) = 0T(\vec{v}) = \vec{0}$.

(b) (5 points) If A is an invertible $n \times n$ matrix, then $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ in $\mathbb{R}^n$ for every $\vec{b}$ in $\mathbb{R}^n$.

Answer: TRUE

Since A is invertible $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$. Thus the solution to $A\vec{x} = \vec{b}$ exists and is unique for every vector $\vec{b}$ in $\mathbb{R}^n$.

(c) (5 points) If $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations such that $S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \vec{x}$ and $T(S(\vec{x})) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \vec{x}$, then T is an invertible transformation.

Answer: FALSE

Note that the matrix of S is invertible because it has determinant 6. If T were invertible, then its matrix would be invertible, so the product of these matrices (the matrix of $T \circ S$) would also be invertible. However, the matrix of $T \circ S$ is not invertible since it has determinant 0. Therefore T can't be invertible.

(d) (5 points) If a system of linear equations has a unique solution, then the system has the same number of variables as equations.

Answer: FALSE

$$\begin{cases} x+y=0 \\ x-y=0 \\ 2x+2y=0 \end{cases}$$ has a unique solution, but it has 2 variables and 3 equations.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

(a) (5 points) Suppose A is the 2×2 matrix of the transformation that rotates vectors in $\mathbb{R}^2$ by an angle θ. Then there is a nonzero vector $\vec{x}$ satisfying the equation $A\vec{x} = -\vec{x}$.

Answer: SOMETIMES

Case when it is true: $\theta = \pi$. Then $A = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\Rightarrow A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = -\vec{x}.$$

Case when it is not true: $\theta = \frac{\pi}{2}$. Then $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \neq \vec{x}$ if $\vec{x}$ is a nonzero vector.

(b) (5 points) If A and B are invertible $n \times n$ matrices, then $rank(A + B) = rank(A) + rank(B)$.

Answer: NEVER

Since A and B are invertible, then rank(A) = rank(B = n

$\Rightarrow$ rank(A) + rank(B) = 2n.

However, A+B is an n x n matrix, so rank(A+B) is at most n.

(c) (5 points) Define a matrix A and a vector $\vec{v}$ in terms of a constant k as follows:

$$A = \begin{bmatrix} k & k+1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & k \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then $A\vec{v}$ is perpendicular to $\vec{v}$. (Recall: Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v} \cdot \vec{w} = 0$.)

Answer: SOMETIMES

$$A\vec{v} = \begin{bmatrix} k & k+1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & k \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} k \\ -6 \\ 2-2k \end{bmatrix}$$

$$A\vec{v} \cdot \vec{v} = \begin{bmatrix} k \\ -6 \\ 2-2k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = k - 2(2-2k) = 0$$

$$\Rightarrow k - 4 + 4k = 0$$

$$5k = 4$$

$$k = \frac{4}{5}$$

If $k = \frac{4}{5}$, then $A\vec{v}$ is perpendicular to $\vec{v}$. If not, they aren't perpendicular.

(d) (5 points) A linear system with 4 equations and 5 variables has a unique solution, provided that its coefficient matrix has rank 4.

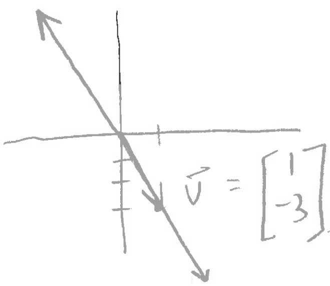
Answer: NEVER

If the coefficient matrix has rank 4 and the system has five variables, then there is one free variable. Consequently, there are infinitely many solutions.

3. (12 points) Let L be the line $y = -3x$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which projects vectors orthogonally onto L:

$$T(\vec{x}) = \text{proj}_L \vec{x}.$$

Find the matrix of T. (Recall: $\text{proj}_L \vec{x} = \left(\dfrac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$, where $\vec{v}$ is any nonzero vector on L.)



$$T(\vec{e_1}) = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -3 \end{bmatrix}}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -3 \end{bmatrix}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ -\frac{3}{10} \end{bmatrix}$$

$$T(\vec{e_2}) = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -3 \end{bmatrix}}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -3 \end{bmatrix}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{-3}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} \\ \frac{9}{10} \end{bmatrix}$$

$\Rightarrow$ matrix of T is $\begin{bmatrix} | & | \\ T(\vec{e_1}) & T(\vec{e_2}) \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{9}{10} \end{bmatrix}$

4. (15 points) Find **all** possible ways of expressing the vector $\vec{e}$ as a linear combination of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, where

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{e} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We must find all solutions to

$$\vec{a}x_1 + \vec{b}x_2 + \vec{c}x_3 + \vec{d}x_4 = \vec{e}.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 2 & 5 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -5 & 3 & -4 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - 5x_3 + 3x_4 = -4 \rightarrow x_1 = -4 + 5x_3 - 3x_4 \\ x_2 + 2x_3 - x_4 = 2. \rightarrow x_2 = 2 - 2x_3 + x_4. \\ x_3, x_4 \text{ free} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 + 5s - 3t \\ 2 - 2s + t \\ s \\ t \end{bmatrix}$$, where s and t are any real numbers.

5. (18 points) (This question has **two** parts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix},$$

(a) Find the inverse transformation T^{-1}, if it exists.

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}.$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 + 2R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

$$\Rightarrow T^{-1}(\vec{x}) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}.$$

(b) Find $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

$$T(\vec{x}) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Multiply both sides by inverse

$$\Rightarrow \vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

6. (15 points) (This question has **two** parts.) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying

$$T\begin{bmatrix}-2\\0\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix} \text{ and } T\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}.$$

(a) Find the matrix of T.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(-\tfrac{1}{2}\begin{bmatrix}-2\\0\end{bmatrix}\right) = -\tfrac{1}{2}T\left(\begin{bmatrix}-2\\0\end{bmatrix}\right) = -\tfrac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}.$$

$$\text{matrix of } T = \begin{bmatrix}1 & -2\\-2 & -1\end{bmatrix}$$

(b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by $\pi/6$ radians. Find the matrix of $T(S^6(\vec{x}))$.

Since S is rotation by $\frac{\pi}{6}$ radians, S^6 is rotation by π

$$\Rightarrow S(\vec{x}) = \begin{bmatrix}\cos\pi & -\sin\pi\\ \sin\pi & \cos\pi\end{bmatrix}\vec{x} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\vec{x}.$$

$$\text{Thus } T(S(\vec{x})) = \begin{bmatrix}1 & -2\\-2 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\vec{x} = \begin{bmatrix}-1 & 2\\2 & 1\end{bmatrix}\vec{x}$$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

DO NOT WRITE ON THIS PAGE.