Northwestern

Math 290-1 Midterm I
Fall Quarter 2018
October 22, 2018

Last name: ____________________ Email address: ____________________

First name: ____________________ NetID: ____________________

Instructions

- Mark your instructor's name.
  - _____ Cañez
  - _____ Newstead
  - _____ Norton
- This examination consists of 10 pages, not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.

   Answer:

   (b) (5 points) If A is an invertible $n \times n$ matrix, then $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ in $\mathbb{R}^n$ for every $\vec{b}$ in $\mathbb{R}^n$.

   Answer:

(c) (5 points) If $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations such that $S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \vec{x}$ and $T(S(\vec{x})) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \vec{x}$, then T is an invertible transformation.

Answer:

(d) (5 points) If a system of linear equations has a unique solution, then the system has the same number of variables as equations.

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

   (a) (5 points) Suppose A is the 2×2 matrix of the transformation that rotates vectors in $\mathbb{R}^2$ by an angle θ. Then there is a nonzero vector $\vec{x}$ satisfying the equation $A\vec{x} = -\vec{x}$.

   Answer:

   (b) (5 points) If A and B are invertible $n \times n$ matrices, then $rank(A + B) = rank(A) + rank(B)$.

   Answer:

(c) (5 points) Define a matrix A and a vector $\vec{v}$ in terms of a constant k as follows:

$$A = \begin{bmatrix} k & k+1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & k \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then $A\vec{v}$ is perpendicular to $\vec{v}$. (Recall: Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v} \cdot \vec{w} = 0$.)

Answer:

(d) (5 points) A linear system with 4 equations and 5 variables has a unique solution, provided that its coefficient matrix has rank 4.

Answer:

3. (12 points) Let L be the line $y = -3x$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which projects vectors orthogonally onto L:

$$T(\vec{x}) = \text{proj}_L \vec{x}.$$

Find the matrix of T. (Recall: $\text{proj}_L \vec{x} = \left(\dfrac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$, where $\vec{v}$ is any nonzero vector on L.)

4. (15 points) Find **all** possible ways of expressing the vector $\vec{e}$ as a linear combination of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, where

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{e} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

5. (18 points) (This question has **two** parts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix},$$

(a) Find the inverse transformation T^{-1}, if it exists.

(b) Find $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

6. (15 points) (This question has **two** parts.) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying

$$T\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \text{ and } T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

(a) Find the matrix of T.

(b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by $\pi/6$ radians. Find the matrix of $T(S^6(\vec{x}))$.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

DO NOT WRITE ON THIS PAGE.