Northwestern University

Math 290-1 Midterm II
Fall Quarter 2018
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Instructions

- Mark your instructor's name.
  - ____ Cañez
  - ____ Newstead
  - ____ Norton
- This examination consists of 10 pages, not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) There exists an invertible linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the x, y-plane.

   Answer: FALSE

   If $\text{im}(T)$ is the x,y-plane, then $\dim(\text{im}(T)) = 2$.
   By Rank-Nullity, $\dim(\ker T) = 3 - 2 = 1$
   But if T is invertible, then $\dim(\ker T) = 0$.
   This is a contradiction. Therefore no such T exists.

   (b) (5 points) There exists a basis $\mathfrak{B}$ of $\mathbb{R}^2$ relative to which we have

   $$\begin{bmatrix} -4 \\ -6 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

   Answer: FALSE

   Since $\begin{bmatrix} -4 \\ -6 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we must have

   $\begin{bmatrix} -4 \\ -6 \end{bmatrix}_B = -2\begin{bmatrix} 2 \\ 3 \end{bmatrix}_B$, but $\begin{bmatrix} -4 \\ -6 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   while $-2\begin{bmatrix} 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$. This is a contradiction. Therefore no such basis exists.

(c) (5 points) Let A be the 2×2 matrix for reflection across a line L in $\mathbb{R}^2$. Then the kernel of $A - I_2$ consists of all vectors in $\mathbb{R}^2$ which are parallel to L. (I_2 denotes the 2×2 identity matrix.)

Answer: TRUE

$\vec{x}$ is in $\ker(A - I_2) \iff (A - I_2)\vec{x} = \vec{0}$

$\iff A\vec{x} = \vec{x}$

Since A is reflection across L, $A\vec{x} = \vec{x}$ if and only if $\vec{x}$ is parallel to L.

(d) (5 points) If k is a scalar, then $\det(kI_3) = k\det(I_3)$. (I_3 denotes the 3×3 identity matrix.)

Answer: FALSE

If $k = 2$, then $\det(2I_3) = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8$

while $2\det(I_3) = 2 \cdot 1 = 2$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

(a) (5 points) For a 4×4 non-identity matrix A, there is a basis $\mathfrak{B}$ of $\mathbb{R}^4$ such that the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$ is I_4. (I_4 denotes the 4×4 identity matrix.)

Answer: NEVER

If the $\mathcal{B}$-matrix of T is I_4, this means $[T(\vec{x})]_{\mathcal{B}} = I_4[\vec{x}]_{\mathcal{B}}$ for all $\vec{x}$ in $\mathbb{R}^4$

That is, $[T(\vec{x})]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}}$ for all $\vec{x}$, ie $T(\vec{x}) = \vec{x}$ for all $\vec{x}$. Thus the standard matrix of T must be I_4. This is a contradiction since we were told that $A \neq I_4$.

(b) (5 points) For a line V through the origin in $\mathbb{R}^3$ and a plane W through the origin in $\mathbb{R}^3$, if $(\vec{v})$ is a basis for V and $(\vec{w}_1, \vec{w}_2)$ is a basis for W, then $(\vec{v}, \vec{w}_1, \vec{w}_2)$ is a basis for $\mathbb{R}^3$.

Answer: SOMETIMES

If $V = \text{span}\{\vec{e}_1\}$ and $W = \text{span}\{\vec{e}_2, \vec{e}_3\}$, then V is a line through $\vec{0}$ with basis $(\vec{e}_1)$, W is a plane through $\vec{0}$ with basis $(\vec{e}_2, \vec{e}_3)$, and $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ form a basis for $\mathbb{R}^3$.

On the other hand, if $V = \text{span}\{\vec{e}_1\}$ and $W = \text{span}\{\vec{e}_1, \vec{e}_2\}$, then $(\vec{e}_1, \vec{e}_1, \vec{e}_2)$ is not a basis for $\mathbb{R}^3$ since it is not linearly independent

(c) (5 points) Let k be a scalar. The determinant of the following matrix is zero.

$$\begin{bmatrix} 0 & 0 & 3k^2+2 & k+3 \\ 0 & 0 & 2k & 0 \\ 2 & 0 & k^3-2k & 3 \\ -1 & -1 & -k^5+1 & 2 \end{bmatrix}$$

expand down this column

Answer: SOMETIMES

$$\det(A) = -1 \begin{vmatrix} 0 & 3k^2+2 & k+3 \\ 0 & 2k & 0 \\ 2 & k^3-2k & 3 \end{vmatrix} = (-1)(2) \begin{vmatrix} 3k^2+2 & k+3 \\ 2k & 0 \end{vmatrix}$$

$$= -2\left(-(k+3)(2k)\right) = 4k^2+12k$$

If $k=0$, $\det(A)=0$.

If $k=1$, $\det(A)=16 \neq 0$

(d) (5 points) If $\vec{v}_1, \ldots, \vec{v}_m$ are linearly **dependent** vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are linearly **dependent**.

Answer: ALWAYS

Since $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent, there exist scalars $c_1, c_2, \ldots, c_m$ (not all zero) such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$. Apply T to both sides of the equations and use the linearity of T to see that $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_mT(\vec{v}_m) = \vec{0}$. Since this is a nontrivial linear relation on $T(\vec{v}_1), \ldots, T(\vec{v}_m)$, this means $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are linearly dependent.

3. (12 points) Let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} 3 \\ k \\ 2 \end{bmatrix} \right\}$. For what values of k is the dimension of V the following:

(a) $\dim(V) = 1$?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & k \\ 4 & k & 2 \end{bmatrix} \xrightarrow{R_3 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & k \\ 0 & k-8 & -10 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & k-8 & -10 \\ 0 & 0 & k \end{bmatrix}$$

Regardless of the value of k, we will always have a leading 1 in the 1^{st} column and either the 2^{nd} or 3^{rd} columns (or both), so $\dim(V)$ is never 1.

(b) $\dim(V) = 2$?

If $k=8$, then the 1^{st} & 3^{rd} columns are lin. independent (and the 2^{nd} is redundant), so $\dim(V) = 2$.

If $k=0$, then the 1^{st} and 2^{nd} columns are lin independent (and the 3^{rd} is redundant), so $\dim(V) = 2$.

(c) $\dim(V) = 3$?

If $k \neq 0, 8$, then there is a leading 1 in each column, so $\dim(V) = 3$.

4. (18 points) Find all vectors $\vec{x}$ in the image of the following matrix **which are also** in its kernel.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

(Hint: You may want to start by finding all solutions to $A^2\vec{x} = \vec{0}$ and figuring out how this relates to the given problem.)

Find all solutions to $A^2\vec{x} = \vec{0}$.

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -4 \\ -1 & 2 & -4 \\ 2 & -4 & 8 \end{bmatrix}$$

$$[A^2 \mid 0] = \left[\begin{array}{ccc|c} -1 & 2 & -4 & 0 \\ -1 & 2 & -4 & 0 \\ 2 & -4 & 8 & 0 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-R_1} \left[\begin{array}{ccc|c} -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_1 + 2x_2 - 4x_3 = 0$

$x_1 = 2x_2 - 4x_3$ $\Rightarrow$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s-4t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}s + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}t$

<u>Note</u>: If $\vec{v}$ is in image of A, this means there exists $\vec{x}$ such that $A\vec{x} = \vec{v}$. But since $\vec{v}$ is in kernel of A, $A\vec{v} = A^2\vec{x} = \vec{0}$. Thus the answer is all vectors of the form $A\vec{x}$ such that $A^2\vec{x} = \vec{0}$

$$A\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}. \quad A\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the answer is all vectors that are in the span of $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, ie span$\left\{\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right\}$.

5. (18 points) Let A be the 2×2 matrix for orthogonal projection onto the line $y = 2x$, and let B be the 2×2 matrix for orthogonal projection onto the line $y = -3x$.

(a) Find a basis $\mathfrak{A}$ of $\mathbb{R}^2$ such that the $\mathfrak{A}$-matrix of $T(\vec{x}) = A\vec{x}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. You do not have to justify your answer.

$$\mathcal{A} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \Rightarrow S_A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

(b) Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of $R(\vec{x}) = B\vec{x}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. You do not have to justify your answer.

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \rightarrow S_B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

(c) Find an invertible 2×2 matrix S satisfying $A = SBS^{-1}$. You **do** have to justify your answer.

We know $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = S_A^{-1} A S_A$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = S_B^{-1} B S_B$.

$$\Rightarrow S_A^{-1} A S_A = S_B^{-1} B S_B \Rightarrow A = S_A S_B^{-1} B S_B S_A^{-1}$$

$$\Rightarrow S = S_A S_B^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

*Solutions may vary, depending on your choice of bases in a) & b).

6. (12 points) Let a and b be some numbers, and let A be the matrix

$$A = \begin{bmatrix} a & b & a & b \\ a & b & a & b \\ a & b & b & a \\ a & b & b & a \end{bmatrix}.$$

(a) Find a basis of the image of A when $a = b$ and neither a nor b is zero.

If $a=b\neq 0$, then all the columns of A are equal to the first column, so a basis for $\operatorname{im}(A)$ is $\left(\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}\right)$.

(b) Find a basis of the image of A when $a = 0$ and $b \neq 0$.

If $a=0$ and $b\neq 0$, then $A = \begin{bmatrix} 0 & b & 0 & b \\ 0 & b & 0 & b \\ 0 & b & b & 0 \\ 0 & b & b & 0 \end{bmatrix}$.

$$\xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 0 & b & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b \\ 0 & 0 & b & -b \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 0 & b & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for $\operatorname{im}(A)$ is $\left(\begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ b \\ b \end{bmatrix}\right)$.

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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