Northwestern University

Math 290-1 Midterm II
Fall Quarter 2018
November 12, 2018

Last name: ________________ Email address: ________________

First name: ________________ NetID: ________________

Instructions

- Mark your instructor's name.

  _____ Cañez

  _____ Newstead

  _____ Norton

- This examination consists of 10 pages, not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This questions has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) There exists an invertible linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the x, y-plane.

   Answer:

   (b) (5 points) There exists a basis $\mathfrak{B}$ of $\mathbb{R}^2$ relative to which we have

$$\begin{bmatrix} -4 \\ -6 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

   Answer:

(c) (5 points) Let A be the 2×2 matrix for reflection across a line L in $\mathbb{R}^2$. Then the kernel of $A - I_2$ consists of all vectors in $\mathbb{R}^2$ which are parallel to L. (I_2 denotes the 2×2 identity matrix.)

Answer:

(d) (5 points) If k is a scalar, then $\det(kI_3) = k\det(I_3)$. (I_3 denotes the 3×3 identity matrix.)

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

   (a) (5 points) For a 4×4 non-identity matrix A, there is a basis $\mathfrak{B}$ of $\mathbb{R}^4$ such that the $\mathfrak{B}$-matrix of the transformation $T(\vec{x}) = A\vec{x}$ is I_4. (I_4 denotes the 4×4 identity matrix.)

   Answer:

   (b) (5 points) For a line V through the origin in $\mathbb{R}^3$ and a plane W through the origin in $\mathbb{R}^3$, if $(\vec{v})$ is a basis for V and $(\vec{w}_1, \vec{w}_2)$ is a basis for W, then $(\vec{v}, \vec{w}_1, \vec{w}_2)$ is a basis for $\mathbb{R}^3$.

   Answer:

(c) (5 points) Let k be a scalar. The determinant of the following matrix is zero.

$$\begin{bmatrix} 0 & 0 & 3k^2+2 & k+3 \\ 0 & 0 & 2k & 0 \\ 2 & 0 & k^3-2k & 3 \\ -1 & -1 & -k^5+1 & 2 \end{bmatrix}.$$

Answer:

(d) (5 points) If $\vec{v}_1, \ldots, \vec{v}_m$ are linearly **dependent** vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are linearly **dependent**.

Answer:

3. (12 points) Let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} 3 \\ k \\ 2 \end{bmatrix} \right\}$. For what values of k is the dimension of V the following:

(a) $\dim(V) = 1$?

(b) $\dim(V) = 2$?

(c) $\dim(V) = 3$?

4. (18 points) Find all vectors $\vec{x}$ in the image of the following matrix **which are also** in its kernel.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

(Hint: You may want to start by finding all solutions to $A^2\vec{x} = \vec{0}$ and figuring out how this relates to the given problem.)

5. (18 points) Let A be the 2×2 matrix for orthogonal projection onto the line $y = 2x$, and let B be the 2×2 matrix for orthogonal projection onto the line $y = -3x$.

   (a) Find a basis $\mathfrak{A}$ of $\mathbb{R}^2$ such that the $\mathfrak{A}$-matrix of $T(\vec{x}) = A\vec{x}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. You do not have to justify your answer.

   (b) Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of $R(\vec{x}) = B\vec{x}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. You do not have to justify your answer.

   (c) Find an invertible 2×2 matrix S satisfying $A = SBS^{-1}$. You **do** have to justify your answer.

6. (12 points) Let a and b be some numbers, and let A be the matrix

$$A = \begin{bmatrix} a & b & a & b \\ a & b & a & b \\ a & b & b & a \\ a & b & b & a \end{bmatrix}.$$

(a) Find a basis of the image of A when $a = b$ and neither a nor b is zero.

(b) Find a basis of the image of A when $a = 0$ and $b \neq 0$.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

DO NOT WRITE ON THIS PAGE.