Northwestern

Math 290-2 Final Exam
Winter Quarter 2019
March 18, 2019

Last name: ____________________ Email address: ____________________

First name: ____________________ NetID: ____________________

Instructions

- Mark your instructor's name.

  _____ Cañez

  _____ Newstead

  _____ Norton

- This examination consists of 16 pages, not including this cover page. Verify that your copy of this examination contains all 16 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 150 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This question has **six** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) Suppose the second-order Taylor polynomial of a C^2 function $f(x, y)$ at $(0, 0)$ is

   $$p_2(x, y) = 4 - 8x + 4x^2 - 3y^2.$$

   Then $(0, 0)$ is a saddle point of f. (Recall that C^2 means that all second order partial derivatives of f exist and are continuous on the domain of f.)

   Answer:

   Solution: FALSE

   Since the coefficient on the x term in p_2 is, by definition, $f_x(0, 0)$, I can tell that $f_x(0, 0) = -8 \neq 0$. Therefore $(0, 0)$ is not a critical point of f. Consequently, it cannot be a saddle point of f.

   (b) (5 points) The surface with equation $\rho = 4 \cos \varphi$ in spherical coordinates is a sphere of radius 2.

   Answer:

   Solution: TRUE

   Converting to Cartesian coordinates, the equation becomes

   $$\begin{aligned} \rho &= 4 \cos \varphi \\ \rho^2 &= 4\rho \cos \varphi \\ x^2 + y^2 + z^2 &= 4z \\ x^2 + y^2 + z^2 - 4z &= 0 \\ x^2 + y^2 + (z^2 - 4z + 4) &= 4 \\ x^2 + y^2 + (z - 2)^2 &= 4. \end{aligned}$$

   Thus the equation is the equation of a sphere with radius 2.

(c) (5 points) Let A be an $n \times n$ matrix, and suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of A such that

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_1 &= 1 \\ \vec{v}_2 \cdot \vec{v}_2 &= 2 \\ &\vdots \\ \vec{v}_n \cdot \vec{v}_n &= n \end{aligned}$$

and $\vec{v}_k \cdot \vec{v}_l = 0$ if $k \neq l$. Then A is symmetric.

Answer:

Solution: TRUE
The given information tells us that the vectors $\vec{v}_1, \ldots, \vec{v}_n$ form an orthogonal set. Dividing each vector by its length would give us a set of orthonormal eigenvectors for A. We know that a matrix is symmetric if and only if it has an orthonormal eigenbasis. Thus A must be symmetric.

(d) (5 points) There exists a real number a such that the matrix $\begin{bmatrix} a^2+1 & -1 \\ 1 & a^2+1 \end{bmatrix}$ is orthogonal.

Answer:

Solution: FALSE
By definition, a matrix is orthogonal if and only if its columns form an orthonormal set. However, the first column has length $\sqrt{(a^2+1)^2+1^2} > 1$ for all a. Thus, since the first column of A does not have length 1, A is not orthogonal.

(e) (5 points) The function $f(x,y,z) = xyz$ has a local maximum at $(0,0,0)$.

Answer:

Solution: FALSE

f has a local max at $(0,0,0)$ if there exists a small ball around $(0,0,0)$ in $\mathbb{R}^3$ such that $f(0,0,0) \geq f(x,y,z)$ for all points in the ball. However, if I start at $(0,0,0)$ and move along the ray $x = y = z$ in the first octant, then $f(x,y,z) = x^3$ along this ray, and f increases as I move away from the origin. Thus f can't have a local max at $(0,0,0)$.

(f) (5 points) There exists a function f of class C^2 such that $\frac{\partial f}{\partial x} = y^3 - 2x$ and $\frac{\partial f}{\partial y} = y - 3xy^2$. (Recall that C^2 means that all second order partial derivatives of f exist and are continuous on the domain of f.)

Answer:

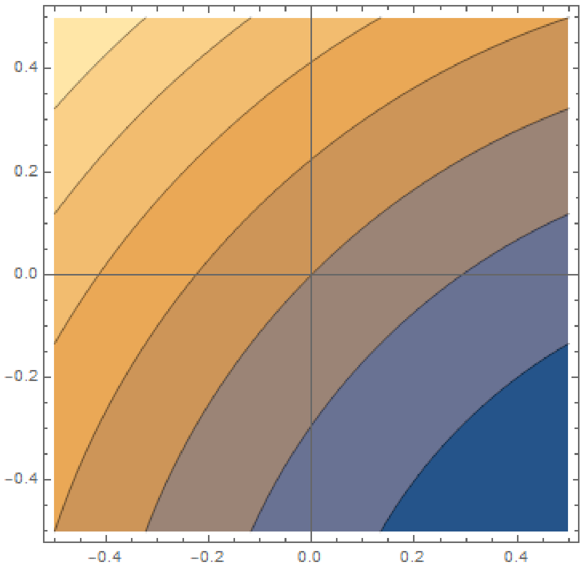
Solution: FALSE

If f is of class C^2, then Clairaut's Theorem must apply to f, i.e., $\dfrac{\partial f^2}{\partial y \partial x} = \dfrac{\partial f^2}{\partial x \partial y}$. However, $\dfrac{\partial f^2}{\partial y \partial x} = \dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial x}) = 3y^2$ while $\dfrac{\partial f^2}{\partial x \partial y} = \dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial y}) = -3y^2$. Thus no such function exists.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **six** parts.)

   (a) (5 points) Consider the surface $z = f(x, y)$ whose level curves are graphed below. The equation $2x + 3y - z = 0$ is the equation of the tangent plane to the surface at $(0, 0, 0)$. (In the graph, lighter colors correspond to larger values of f.)

   Answer:



Solution: NEVER

We can tell from the graph that $f_x(0,0) < 0$ while $f_y(0,0) > 0$. Since the normal vector to the tangent plane to f at $(0,0)$ is $\begin{bmatrix} f_x(0,0) \\ f_y(0,0) \\ -1 \end{bmatrix}$, the equation of the tangent plane should have coefficients of opposite sign on x and y. Since it doesn't, the given equation cannot be the equation of the tangent plane to f at the origin.

   (b) (5 points) Let k be a constant. The function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ k & \text{if } (x, y) = (0, 0) \end{cases}$$

   is continuous on all of $\mathbb{R}^2$.

   Answer:

Solution: NEVER

Note that along the x-axis, $f(x, y) = f(x, 0) = 0$. Thus if we approach $(0, 0)$ along the x-axis, i.e., $y = 0$, $x \to 0$, then the function values approach 0. However, along the line $x = y$, $f(x, y) = f(x, x) =$

$\frac{x^2}{2x^2}$. Thus if we approach $(0,0)$ along this line, i.e., $x = y \to 0$, then the function values approach $\frac{1}{2}$. Since the function approaches different values along different paths toward $(0,0)$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Therefore there is no value of k that will make f continuous at $(0,0)$.

(c) (5 points) Let $f(x, y)$ be a function that is continuous at all points in $\mathbb{R}^2$, and suppose that $f(1, 0)$ is the maximum value that f attains on the disc $x^2 + y^2 \leq 1$. Then $f(1, 0)$ is the absolute maximum value of f on all of $\mathbb{R}^2$.

Solution: SOMETIMES

If $f(x, y) = x^2 + y^2$, then the maximum value of $f(x, y)$ on $x^2 + y^2 \leq 1$ is 1 and it is attained everywhere along the boundary, including at $(1, 0)$. However, $f(1, 0) = 1$ is not the absolute max of the function, since, for example, $f(3, 0) = 9 > 1$.

If $f(x, y) = -(x - 1)^2 - y^2$, then the maximum value of $f(x, y)$ on $x^2 + y^2 \leq 1$ is 0, and it is attained at $(1, 0)$. $f(1, 0) = 0$ is also the absolute max of the function, since f is an elliptic paraboloid that opens down and has its vertex at $(1, 0)$.

Answer:

(d) (5 points) Consider a subspace W of $\mathbb{R}^4$ with basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be the result of applying the Gram–Schmidt process to the basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$ of W, and let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the result of applying the Gram–Schmidt process to the basis $\vec{w}_2, \vec{w}_1, \vec{w}_3$ of W. Then $\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

Answer:

Solution: ALWAYS

By construction in the Gram-Schmidt process, $\vec{u}_1$ is a scalar multiple of $\vec{w}_1$ and $\vec{u}_2$ is in $\text{span}\{\vec{w}_1, \vec{w}_2\}$. Thus $\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{w}_1, \vec{w}_2\}$. Similarly, $\vec{v}_1$ is a scalar multiple of $\vec{w}_2$ and $\vec{v}_2$ is in $\text{span}\{\vec{w}_2, \vec{w}_1\}$. Thus $\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{w}_1, \vec{w}_2\}$. So $\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

(e) (5 points) Suppose $f(x, y)$ is a differentiable function which attains a local maximum at $(0, 0)$ subject to the constraint $g(x, y) = c$, where $g(x, y)$ is differentiable and both $\nabla f(0, 0)$ and $\nabla g(0, 0)$ are nonzero. Then there is a scalar λ satisfying $\nabla g(0, 0) = \lambda \nabla f(0, 0)$.

Answer:

Solution: ALWAYS

By the Lagrange multipliers theorem, we know that since f attains a local max at $(0, 0)$ subject to the constraint $g(x, y) = c$ and $\nabla g(0, 0) \neq \vec{0}$, then there must exist a scalar μ such that $\nabla f(0, 0) = \mu \nabla g(0, 0)$. Moreover, since both gradients are nonzero at the origin, μ cannot be zero. Thus, dividing both sides by μ we get that $\nabla g(0, 0) = \frac{1}{\mu} \nabla f(0, 0)$. Thus there exists a scalar $\lambda = \frac{1}{\mu}$ such that the given equality is satisfied.

(f) (5 points) Let A be a 5×5 matrix such that $\det(A - tI_5) = t^2(t - 1)^2(t + 1)$ for all real t. Then $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^5$.

Answer:

Solution: NEVER

Since 0 is a root of $\det(A - tI_5)$, 0 is an eigenvalue of A. Thus there exists a (nonzero) eigenvector $\vec{v}$ of A such that $A\vec{v} = \vec{0}$. For this vector, $\|A\vec{v}\| \neq \|\vec{v}\|$.

3. (15 points) Find and classify all the critical points of the function

$$f(x, y) = x^2 - y^3 - x^2y + y.$$

Solution: $f_x(x, y) = 2x - 2xy$ and $f_y(x, y) = -3y^2 - x^2 + 1$. Since f_x and f_y are defined for all points, the only critical points of f are the points that satisfy both $f_x = 0$ and $f_y = 0$. $f_x(x, y, z) = 0$ implies either $x = 0$ or $y = 1$. Plugging $x = 0$ into the equation $f_y(x, y, z) = 0$ gives $y = \pm\sqrt{1/3}$. Plugging $y = 1$ into $f_y(x, y, z) = 0$ gives $x^2 = -2$, which yields no solutions. Thus the only critical points are $(0, \sqrt{1/3})$ and $(0, -\sqrt{1/3})$.

We use the second derivative test to classify the critical points.

$$Hf(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 - 2y & -2x \\ -2x & -6y \end{bmatrix}$$

$$Hf(0, \sqrt{1/3}) = \begin{bmatrix} 2 - 2/\sqrt{3} & 0 \\ 0 & -6/\sqrt{3} \end{bmatrix}$$

The eigenvalues of $Hf(0, \sqrt{1/3})$ are $2-2/\sqrt{3}$ and $-6/\sqrt{3}$. Since one is negative and the other is positive, we conclude that f has a saddle point at $(0, \sqrt{1/3})$.

$$Hf(0, -\sqrt{1/3}) = \begin{bmatrix} 2 + 2/\sqrt{3} & 0 \\ 0 & 6/\sqrt{3} \end{bmatrix}$$

The eigenvalues of $Hf(0, -\sqrt{1/3})$ are $2+2/\sqrt{3}$ and $6/\sqrt{3}$. Since both eigenvalues are positive, we conclude that f has a local min at $(0, -\sqrt{1/3})$.

4. Let V be the plane in $\mathbb{R}^3$ described by the equation $2x - y + z = 0$.

(a) (9 points) Find an orthonormal basis $\vec{u}_1, \vec{u}_2$ in $\mathbb{R}^3$ of the plane.

Solution: Any two linearly independent vectors that are orthogonal to the normal vector $\vec{n} = (2, -1, 1)$ form a basis for the plane. Let $\vec{v}_1 = (0, 1, 1)$ and let $\vec{v}_2 = (1, 2, 0)$. Apply Gram-Schmidt to $\vec{v}_1, \vec{v}_2$. Compute

$$\vec{x}_2 = \vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2 = (1, 2, 0) - 2/2(0, 1, 1) = (1, 1, -1).$$

The $\vec{u}_1 = \frac{1}{\sqrt{2}}(0, 1, 1)$ and $\vec{u}_2 = \frac{1}{\sqrt{3}}(1, 1, -1)$ is an orthonormal basis for the plane.

(b) (7 points) Find a vector $\vec{u}_3$ in $\mathbb{R}^3$ such that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is an orthonormal basis of $\mathbb{R}^3$.

Solution: The normal vector to the plane is perpendicular to both $\vec{u}_1$ and $\vec{u}_2$, so we can take $\vec{u}_3 = \frac{1}{\sqrt{5}}(2, -1, 1)$.
Alternatively, we can compute

$$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\vec{i} + \vec{j} - \vec{k}$$

and define $\vec{u}_3 = 1/\sqrt{5}(-2, 1, -1)$.

(c) (5 points) Find the matrix of orthogonal projection onto V. You may express your answer as a product of two matrices which you do not have to multiply out.

Solution:

$$A = QQ^T, \text{ where } Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

5. Suppose the temperature at a point (x, y) on a hot tin roof is given by $T(x, y)$, where $T : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. Suppose a cat is on the roof at the point $(2, 5)$. He notices that if he moves in the direction of $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the temperature of the roof increases at a rate of 3 deg/m, and if he moves in the direction of $\vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the temperature decreases at a rate of 2 deg/m.

(a) (7 points) In what direction should the cat move if he wants to cool down most rapidly?

Solution: From the information given we know $D_{\vec{u}}T(2, 5) = T_x(2, 5) \cdot 0 + T_y(2, 5) \cdot 1 = 3$. So $T_y(2, 5) = 3$. Also, $D_{\vec{v}}T(2, 5) = T_x(2, 5) \cdot \frac{2}{\sqrt{5}} + T_y(2, 5) \cdot \frac{1}{\sqrt{5}} = -2$, so $T_x(2, 5) = -\sqrt{5} - \frac{3}{2}$. Thus the direction the cat should walk in to cool down most rapidly is $-\nabla T(2, 5) = \begin{bmatrix} \sqrt{5} + \frac{3}{2} \\ -3 \end{bmatrix}$.

(b) (7 points) If the cat moves in the direction of $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, will the temperature of the roof increase or decrease? Justify your answer.

Solution: Let $\vec{w} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $D_{\vec{w}}T(2, 5) = \vec{w} \cdot \nabla T(2, 5) = \frac{1}{\sqrt{2}} \cdot (-\sqrt{5} - \frac{3}{2}) + \frac{1}{\sqrt{2}} \cdot 3 = -\frac{\sqrt{5}}{\sqrt{2}} - \frac{3}{2\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{3-2\sqrt{5}}{2\sqrt{2}} < 0$. So the temperature decreases in that direction.

6. (15 points) Let D be the region in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying the inequality $x^2+y^2+z^2 \leq 4$. Find the absolute maximum and minimum values of $f(x, y, z) = xy + z$ on D.

Solution: $\nabla f(x, y, z) = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$. Thus there are no points where the partial derivatives are undefined and no points where all partial derivatives are 0, so there are no critical points.

To find the extrema of f on the boundary $g(x, y, z) = x^2 + y^2 + z^2 = 4$, we use Lagrange multipliers.

$$\nabla g(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

The only point where $\nabla g = \vec{0}$ is $(0, 0, 0)$, which doesn't satisfy the constraint condition. Lastly, we find the points that satisfy $\nabla f = \lambda \nabla g$ for some scalar λ:

$$\begin{aligned} y &= 2x\lambda \\ x &= 2y\lambda \\ 1 &= 2z\lambda \\ x^2 + y^2 + z^2 &= 4 \end{aligned}$$

First assume x, y, z are all nonzero. Solving each equation for λ we get $\frac{y}{2x} = \frac{x}{2y} = \frac{1}{2z}$. The equation $\frac{y}{2x} = \frac{x}{2y}$ gives $x = \pm y$. The equation $\frac{y}{2x} = \frac{1}{2z}$ gives $z = \frac{x}{y} = \pm 1$. Plugging into the constraint we get $x^2 + x^2 + 1 = 4 \implies x = \pm\sqrt{\frac{3}{2}}$. Therefore, the points that satisfy $\nabla f = \lambda \nabla g$ for some λ are $(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1), (\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}, -1), (-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, -1)$, and $(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}, 1)$. f is largest at the points where x and y have the same sign. At these points it equals $\frac{5}{2}$. f is smallest where x and y have opposite signs. At these points it equals $-\frac{5}{2}$

7. (10 points) The cost of producing a batch of peanut butter cups is given by the equation $Q(x, y) = 170 + 50x^2 + 5xy$, where x is the cost of peanut butter and y is the cost of chocolate required for the batch. It is estimated that, in 6 months, the cost of peanut butter will be \$9/batch and increasing at a rate of \$0.50/month, while the cost of chocolate will be \$10 and increasing at a rate of \$0.20/month. At what rate will the cost of producing a batch of peanut butter cups be changing with respect to time 6 months from now?

Solution: We are given that, in 6 months, $x = 9$, $dx/dt = .5$, $y = 10$, and $dy/dt = .2$. By the chain rule,

$$\begin{aligned}\frac{dQ}{dt} &= \frac{\partial Q}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Q}{\partial y} \cdot \frac{dy}{dt} \\ &= (100x + 5y)\frac{dx}{dt} + (5x)\frac{dy}{dt} \\ &= (100 \cdot 9 + 5 \cdot 10)(.5) + (5 \cdot 9)(.2) \\ &= 484\end{aligned}$$

Thus the rate of change of Q with respect to t in 6 months is \$484/month.

8. (15 points) Four Math 290-2 students took part in a study to test the correlation between bedtime and coffee consumption. Each student reported what time they went to bed and how much coffee they consumed the next day. The following table summarises the findings of the study.

	Student 1	Student 2	Student 3	Student 4
Bedtime (hours after midnight)	-2	-1	0	3
Coffee consumed the next day (fl oz)	0	9	31	50

Use the method of least squares to find the function of the form $f(t) = a + bt$ that best fits this data, where $f(t)$ represents the amount of coffee (in fluid ounces) consumed by a student whose bedtime is t hours after midnight.

Solution: We compute the least squares solution (a^*, b^*) to the system

$$\begin{cases} a - 2b = 0 \\ a - b = 9 \\ a = 31 \\ a + 3b = 50 \end{cases}$$

Let $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 3 \end{pmatrix}$ and let $\mathbf{b} = \begin{pmatrix} 0 \\ 9 \\ 31 \\ 50 \end{pmatrix}$. Then

$$A^T A = \begin{pmatrix} 1+1+1+1 & -2-1+0+3 \\ -2-1+0+3 & 4+1+0+9 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix} = 2\begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

and so $(A^T A)^{-1} = \frac{1}{2} \cdot \frac{1}{14} \cdot \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}$. And

$$A^T\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 9 \\ 31 \\ 50 \end{pmatrix} = \begin{pmatrix} 0+9+31+50 \\ 0-9+0+150 \end{pmatrix} = \begin{pmatrix} 90 \\ 141 \end{pmatrix}$$

So the least squares solution is given by

$$\begin{pmatrix} a^* \\ b^* \end{pmatrix} = (A^T A)^{-1} A^T \mathbf{b} = \frac{1}{28}\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 90 \\ 141 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 630 \\ 282 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 315 \\ 141 \end{pmatrix} = \begin{pmatrix} 45/2 \\ 141/14 \end{pmatrix}$$

So the function f is given by $\boxed{f(t) = \frac{45}{2} + \frac{141}{14}t}$.

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