Northwestern

Math 290-2 Final Exam
Winter Quarter 2019
March 18, 2019

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- Mark your instructor's name.

  _____ Cañez

  _____ Newstead

  _____ Norton

- This examination consists of 15 pages, not including this cover page. Verify that your copy of this examination contains all 15 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 150 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This question has **six** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) Suppose the second-order Taylor polynomial of a C^2 function $f(x, y)$ at $(0, 0)$ is

   $$p_2(x, y) = 4 - 8x + 4x^2 - 3y^2.$$

   Then $(0, 0)$ is a saddle point of f. (Recall that C^2 means that all second order partial derivatives of f exist and are continuous on the domain of f.)

   Answer:

   (b) (5 points) The surface with equation $\rho = 4\cos\varphi$ in spherical coordinates is a sphere of radius 2.

   Answer:

(c) (5 points) Let A be an $n \times n$ matrix, and suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of A such that

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_1 &= 1 \\ \vec{v}_2 \cdot \vec{v}_2 &= 2 \\ &\vdots \\ \vec{v}_n \cdot \vec{v}_n &= n \end{aligned}$$

and $\vec{v}_k \cdot \vec{v}_l = 0$ if $k \neq l$. Then A is symmetric.

Answer:

(d) (5 points) There exists a real number a such that the matrix $\begin{bmatrix} a^2+1 & -1 \\ 1 & a^2+1 \end{bmatrix}$ is orthogonal.

Answer:

(e) (5 points) The function $f(x,y,z) = xyz$ has a local maximum at $(0,0,0)$.

Answer:

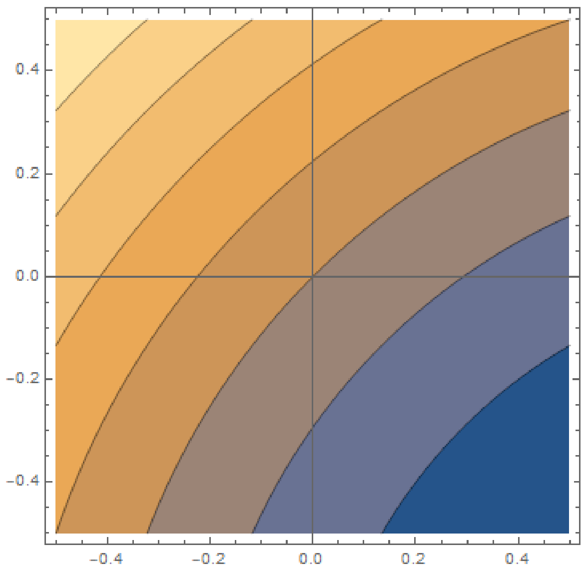
(f) (5 points) There exists a function f of class C^2 such that $\frac{\partial f}{\partial x} = y^3 - 2x$ and $\frac{\partial f}{\partial y} = y - 3xy^2$. (Recall that C^2 means that all second order partial derivatives of f exist and are continuous on the domain of f.)

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **six** parts.)

   (a) (5 points) Consider the surface $z = f(x, y)$ whose level curves are graphed below. The equation $2x + 3y - z = 0$ is the equation of the tangent plane to the surface at $(0, 0, 0)$. (In the graph, lighter colors correspond to larger values of f.)

   Answer:



   (b) (5 points) Let k be a constant. The function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ k & \text{if } (x,y) = (0,0) \end{cases}$$

   is continuous on all of $\mathbb{R}^2$.

   Answer:

(c) (5 points) Let $f(x, y)$ be a function that is continuous at all points in $\mathbb{R}^2$, and suppose that $f(1, 0)$ is the maximum value that f attains on the disc $x^2 + y^2 \leq 1$. Then $f(1, 0)$ is the absolute maximum value of f on all of $\mathbb{R}^2$.

Answer:

(d) (5 points) Consider a subspace W of $\mathbb{R}^4$ with basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be the result of applying the Gram–Schmidt process to the basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$ of W, and let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the result of applying the Gram–Schmidt process to the basis $\vec{w}_2, \vec{w}_1, \vec{w}_3$ of W. Then $\operatorname{span}\{\vec{u}_1, \vec{u}_2\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

Answer:

(e) (5 points) Suppose $f(x, y)$ is a differentiable function which attains a local maximum at $(0, 0)$ subject to the constraint $g(x, y) = c$, where $g(x, y)$ is differentiable and both $\nabla f(0, 0)$ and $\nabla g(0, 0)$ are nonzero. Then there is a scalar λ satisfying $\nabla g(0, 0) = \lambda \nabla f(0, 0)$.

Answer:

(f) (5 points) Let A be a 5×5 matrix such that $\det(A - tI_5) = t^2(t - 1)^2(t + 1)$ for all real t. Then $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^5$.

Answer:

3. (15 points) Find and classify all the critical points of the function

$$f(x, y) = x^2 - y^3 - x^2y + y.$$

4. Let V be the plane in $\mathbb{R}^3$ described by the equation $2x - y + z = 0$.
   (a) (9 points) Find an orthonormal basis $\vec{u}_1, \vec{u}_2$ in $\mathbb{R}^3$ of the plane.

   (b) (7 points) Find a vector $\vec{u}_3$ in $\mathbb{R}^3$ such that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is an orthonormal basis of $\mathbb{R}^3$.

   (c) (5 points) Find the matrix of orthogonal projection onto V. You may express your answer as a product of two matrices which you do not have to multiply out.

5. Suppose the temperature at a point (x, y) on a hot tin roof is given by $T(x, y)$, where $T : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. Suppose a cat is on the roof at the point $(2, 5)$. He notices that if he moves in the direction of $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the temperature of the roof increases at a rate of 3 deg/m, and if he moves in the direction of $\vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the temperature decreases at a rate of 2 deg/m.

   (a) (7 points) In what direction should the cat move if he wants to cool down most rapidly?

   (b) (7 points) If the cat moves in the direction of $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, will the temperature of the roof increase or decrease? Justify your answer.

6. (15 points) Let D be the region in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying the inequality $x^2+y^2+z^2 \leq 4$. Find the absolute maximum and minimum values of $f(x, y, z) = xy + z$ on D.

7. (10 points) The cost of producing a batch of peanut butter cups is given by the equation $Q(x, y) = 170 + 50x^2 + 5xy$, where x is the cost of peanut butter and y is the cost of chocolate required for the batch. It is estimated that, in 6 months, the cost of peanut butter will be \$9/batch and increasing at a rate of \$0.50/month, while the cost of chocolate will be \$10 and increasing at a rate of \$0.20/month. At what rate will the cost of producing a batch of peanut butter cups be changing with respect to time 6 months from now?

8. (15 points) Four Math 290-2 students took part in a study to test the correlation between bedtime and coffee consumption. Each student reported what time they went to bed and how much coffee they consumed the next day. The following table summarises the findings of the study.

	Student 1	Student 2	Student 3	Student 4
Bedtime (hours after midnight)	-2	-1	0	3
Coffee consumed the next day (fl oz)	0	9	31	50

Use the method of least squares to find the function of the form $f(t) = a + bt$ that best fits this data, where $f(t)$ represents the amount of coffee (in fluid ounces) consumed by a student whose bedtime is t hours after midnight.

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