Northwestern

Math 290-2 Midterm 1
Winter Quarter 2019
February 4, 2019

Last name: Solutions ______ Email address: ______

First name: ______ NetID: ______

Instructions

- Mark your instructor's name.

  ____ Cañez

  ____ Newstead

  ____ Norton

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This question has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) There exist <u>nonzero</u> vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ such that $\vec{a} \times \vec{b} = \vec{a}$.

   Answer: FALSE

   Since $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then if $\vec{a} \times \vec{b} = \vec{a}$, $\vec{a}$ must be perpendicular to itself, ie $\vec{a} \cdot \vec{a} = 0$. This is only true if $\vec{a} = \vec{0}$, which is not the case.

   (b) (5 points) Let A be an $n \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. If $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$, then $A^T A$ is diagonal.

   Answer: TRUE

$$A^T A = \begin{bmatrix} — \vec{v}_1^T — \\ \vdots \\ — \vec{v}_n^T — \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1^T \vec{v}_1 & \vec{v}_1^T \vec{v}_2 & \cdots & \vec{v}_1^T \vec{v}_n \\ \vdots & & & \\ \vec{v}_n^T \vec{v}_1 & \vec{v}_n^T \vec{v}_2 & \cdots & \vec{v}_n^T \vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \|\vec{v}_1\|^2 & & & 0 \\ & \|\vec{v}_2\|^2 & & \\ & & \ddots & \\ 0 & & & \|\vec{v}_n\|^2 \end{bmatrix}$$

(c) (5 points) Orthogonal projection onto a plane is an orthogonal transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Answer: FALSE

A transformation T is orthogonal if and only if $\|T\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}$ in $\mathbb{R}^3$. Since orthogonal projection sends nonzero vectors to the zero vector, it is not an orthogonal transformation.

(d) (5 points) There exists a 2×2 orthogonal matrix Q such that $Q^T \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

Answer: FALSE

$Q^T \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = Q \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} Q^T$.

Since Q is orthogonal and $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonal, $Q \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} Q^T$ is a symmetric matrix. However, $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ is not symmetric

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

(a) (5 points) Let A be a 3×3 matrix. For all $\vec{x}$ in $\mathbb{R}^3$, $\vec{x} \cdot A\vec{x} = 3x^{10} + 4xy - 6yz + 2z^3$.

Answer: NEVER

For any 3×3 matrix A,

$$\bar{x} \cdot A\bar{x} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}$$

$$= x(a_{11}x + a_{12}y + a_{13}z) + y(a_{21}x + a_{22}y + a_{23}z) + z(a_{31}x + a_{32}y + a_{33}z)$$

So $\bar{x} \cdot A\bar{x}$ can't have a term of degree 10.

(b) (5 points) Let A be a 2×2 orthogonal matrix, and let $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $S^{-1}AS$ is an orthogonal matrix.

Answer: SOMETIMES

If $A = I_2$, then $S^{-1}AS = I$ is orthogonal

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $S^{-1}AS = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

is not orthogonal because its columns aren't orthonormal.

(c) (5 points) Suppose A is an $m \times n$ matrix and $\vec{b}$ is a vector in $\mathbb{R}^m$. If $\vec{b}$ is perpendicular to every column of A, then the zero vector in $\mathbb{R}^n$ is a least squares solution to $A\vec{x} = \vec{b}$.

Answer: ALWAYS

By definition, the least squares solution $\vec{x}^*$ of the equation $A\vec{x} = \vec{b}$ satisfies $A\vec{x}^* = \text{proj}_{\text{im}A}\vec{b}$.

Since $\vec{b}$ is perpendicular to every column of A, $\text{proj}_{\text{im}A}\vec{b} = \vec{0}$. Thus $\vec{x}^* = \vec{0}$ satisfies

$A\vec{x}^* = \text{proj}_{\text{im}A}\vec{b} = \vec{0}$

(d) (5 points) Let k be a real number and let A be a 3×3 matrix with eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ k \end{bmatrix}$$

corresponding to eigenvalues $1, 1, 2$, respectively. Then A is symmetric.

Answer: SOMETIMES

linearly independent

A is symmetric $\Leftrightarrow$ every vector in $E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is orthogonal to every vector in $E_2 = \text{span}\left\{\begin{bmatrix} 3 \\ 0 \\ k \end{bmatrix}\right\}$.

$\Leftrightarrow \begin{bmatrix} 3 \\ 0 \\ k \end{bmatrix}$ is orthogonal to both $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

$\Leftrightarrow \begin{bmatrix} 3 \\ 0 \\ k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3 - k = 0$ and $\begin{bmatrix} 3 \\ 0 \\ k \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3 + k = 0$.

$\Leftrightarrow k = 3$

3. (15 points) Find the point(s) on the curve $6x^2 + 4xy + 3y^2 = 1$ which are closest to $(0, 0)$.

Diagonalize the quadratic form:

$$6x^2 + 4xy + 3y^2 = [x\ y]\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues

$$\det(A - \lambda I) = 0 \Rightarrow (6-\lambda)(3-\lambda) - 4 = 0$$

$$18 - 9\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 9\lambda + 14 = 0$$

$$(\lambda - 7)(\lambda - 2) = 0$$

$$\lambda = 2, 7.$$

Eigenvectors

$\lambda = 2$: $[A - 2I \,|\, \vec{0}] = \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 + x_2 = 0 \Rightarrow x_1 = -\frac{1}{2}x_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 7$: $[A - 7I \,|\, \vec{0}] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$6x^2 + 4xy + 3y^2 = 1$ can be rewritten as $2c_1^2 + 7c_2^2 = 1$.
Thus $c_1 = 0, c_2 = \pm\frac{1}{\sqrt{7}}$ are the $\mathcal{B}$ coordinates of the closest points on the curve to $(0,0)$. Points: $\pm\frac{1}{\sqrt{7}} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$,
i.e. $\left(\frac{2}{\sqrt{35}}, \frac{1}{\sqrt{35}}\right), \left(\frac{-2}{\sqrt{35}}, \frac{-1}{\sqrt{35}}\right)$

4. (12 points) Find the function of the form $f(t) = c_0 + c_1 t$ that best fits the data points $(0,-3), (1,-3)$, and $(1,-6)$.

Plug in points to get system of equations:

$c_0 = -3$

$c_0 + c_1 = -3$

$c_0 + c_1 = -6$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix}$$

Least squares soln to $A\vec{x} = \vec{b}$ is $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix}$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -12 \\ -9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -\frac{3}{2} \end{bmatrix}$$

$\Rightarrow f(t) = -3 - \frac{3}{2}t$

5. (15 points) Consider the intersecting lines $l_1 : x = 1+t, y = 6+5t, z = -2t$ and $l_2 : x = -3+3t, y = t, z = 1+t$. Find parametric equations for the line that intersects l_1 and l_2 at their point of intersection and is perpendicular to both l_1 and l_2.

Point of intersection

$1 + t = -3 + 3s$

$6 + 5t = s$

$-2t = 1 + s$

$\xrightarrow{\text{Sub in}}$ $1 + t = -3 + 3(6 + 5t)$

$1 + t = -3 + 18 + 15t$

$-14t = 14$

$t = -1.$

$\Rightarrow s = 6 + 5(-1) = 1$

point:

$x = 1 + (-1) = 0$

$y = 6 + 5(-1) = 1$

$z = -2(-1) = 2$

Direction of line

Since the line must be perpendicular to l_1 and l_2, we take the cross product of the vectors parallel to l_1 and l_2 to find a vector parallel to our line:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_1 \times \vec{v}_2 = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -2 \\ 3 & 1 & 1 \end{bmatrix} = 7\vec{i} - 7\vec{j} - 14\vec{k}$

Line

$x = 0 + 7t$

$y = 1 - 7t$

$z = 2 - 14t$

6. (18 points) Find an orthonormal basis for $V = (\text{im } A)^\perp$, where A is the 4×2 matrix defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \\ -1 & -1 \end{bmatrix}.$$

(Recall: If W is a subspace of $\mathbb{R}^n$, then $W^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } W\}$.)

$\vec{x}$ is in $(\text{im} A)^\perp \iff \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0$ and $\vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} = 0$

$\iff$ $1x_1 + 2x_3 - x_4 = 0$
$1x_2 + 3x_3 - x_4 = 0.$

Solve: $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right] \Rightarrow$ $x_1 = -2x_3 + x_4$
$x_2 = -3x_3 + x_4$

$$\vec{x} = \begin{bmatrix} -2s + t \\ -3s + t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} t$$

$$(\text{im} A)^\perp = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Gram-Schmidt

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} - \dfrac{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} - \dfrac{-5}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \\ \frac{5}{3} \end{bmatrix} \xrightarrow{\times 3} \begin{bmatrix} -1 \\ -4 \\ 3 \\ 5 \end{bmatrix}$

Orthonormal basis

$$\left\{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{51}} \begin{bmatrix} -1 \\ -4 \\ 3 \\ 5 \end{bmatrix}\right\}$$

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