Northwestern

Math 290-2 Midterm 2
Winter Quarter 2019
March 4, 2019

Last name: Solutions ____________ Email address: ____________

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Instructions

- Mark your instructor's name.

  ____ Cañez

  ____ Newstead

  ____ Norton

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This question has **four** parts. Remember: A statement is "true" if it is always true. If not, it is "false.")

   (a) (5 points) The surface in $\mathbb{R}^3$ defined by $x^2 + 2yz = 1$ is a hyperboloid of one sheet.

   Answer: TRUE

$$x^2 + 2yz = \begin{bmatrix} x & y & z \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A.} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0 \Rightarrow \lambda = 1, 1, -1$$

Two positive eigenvalues and one negative eigenvalue implies the surface is a hyperboloid of one sheet.

   (b) (5 points) The surface in $\mathbb{R}^3$ defined in cylindrical coordinates by $z^2 = 1 + r^2\cos(2\theta)$ is an ellipsoid. (Recall: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.)

   Answer: FALSE

$$z^2 = 1 + r^2\cos(2\theta)$$

$$z^2 = 1 + r^2(\cos^2\theta - \sin^2\theta)$$

$$z^2 = 1 + r^2\cos^2\theta - r^2\sin^2\theta$$

$$z^2 = 1 + x^2 - y^2$$

$-x^2 + y^2 + z^2 = 1$. is not an ellipsoid because it has 2 positive coefficients and one negative coefficient. An ellipsoid must have all positive coefficients.

(c) (5 points) The function

$$f(x,y) = \begin{cases} e^{xy} + \frac{x^5}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Answer: TRUE

First compute $\lim_{(x,y)\to(0,0)} \frac{x^5}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^5 \cos^5\theta}{r^2} = \lim_{r\to 0^+} r^3\cos^5\theta$

Note: $-r^3 \leq r^3\cos^5\theta \leq r^3$ for all θ since $-1 \leq \cos^5\theta \leq 1$ and $\lim_{r\to0^+} -r^3 = \lim_{r\to 0^+} r^3 = 0$. Thus by Squeeze Theorem $\lim_{r\to0^+} r^3\cos^5\theta = 0$.

Now compute $\lim_{(x,y)\to(0,0)} e^{xy} + \frac{x^5}{x^2+y^2} = \lim_{(x,y)\to(0,0)} e^{xy} + \lim_{(x,y)\to(0,0)} \frac{x^5}{x^2+y^2}$
$= 1 + 0 = 1 = f(0,0)$

$\Rightarrow f$ is continuous at $(0,0)$.

(d) (5 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(3,0)$ and $\nabla f(3,0) = (1,2)$. If $\vec{x}(t) = (3\cos t, 4\sin t)$, then $D(f\circ\vec{x})(0) = 8$.

Answer: TRUE

By Chain Rule, $D(f\circ\vec{x})(0) = Df(\vec{x}(0))\, D\vec{x}(0)$.

$Df(\vec{x}(0)) = Df(3,0) = \nabla f(3,0)^T = [1 \ 2]$.

$D\vec{x}(t) = \begin{bmatrix} -3\sin t \\ 4\cos t \end{bmatrix} \Rightarrow D\vec{x}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

$\Rightarrow D(f\circ\vec{x})(0) = [1 \ 2]\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 8$ ✓

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This question has **four** parts.)

(a) (5 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable. Then $Df(0,0) = Df(0,0)^T$.

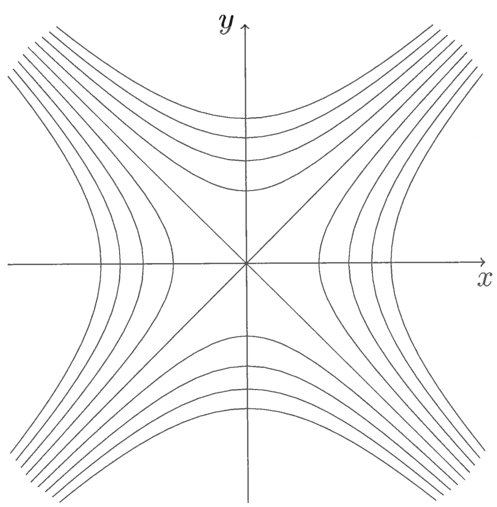
Answer: SOMETIMES.

If $f(x,y) = (0,0)$, then $Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Df(0,0)^T$

If $f(x,y) = (0,x)$, then $Df(x,y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq Df(0,0)^T$

(b) (5 points) The following diagram represents the level curves of the surface $z = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix}$, where A is a 2×2 symmetric matrix with positive determinant.



Answer: NEVER.

Since A is symmetric, $z = [x\ y] A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 c_1^2 + \lambda_2 c_2^2$, where λ_1 and λ_2 are the e-values of A and c_1 and c_2 are the coordinates of $\begin{bmatrix} x \\ y \end{bmatrix}$ with respect to the eigenbasis. The level curves will be hyperbolas if and only if λ_1 and λ_2 have opposite signs. But that implies $\det(A) = \lambda_1 \lambda_2 < 0$, which contradicts the information we were given.

(c) (5 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\vec{a} \in \mathbb{R}^2$, and suppose there exist linearly independent unit vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ such that $D_{\vec{u}} f(\vec{a}) = D_{\vec{v}} f(\vec{a}) = 0$. If $\vec{w}$ is a unit vector in $\mathbb{R}^2$, then $D_{\vec{w}} f(\vec{a}) = 0$.

Answer: ALWAYS

Let $\vec{w}$ be a unit vector in $\mathbb{R}^2$. Since $\vec{u}$ and $\vec{v}$ are linearly independent vectors in $\mathbb{R}^2$, they span $\mathbb{R}^2$. Therefore there exist scalars c_1 and c_2 such that $\vec{w} = c_1\vec{u} + c_2\vec{v}$. Since f is differentiable at $\vec{a}$,

$$D_{\vec{w}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{w} = \nabla f(\vec{a}) \cdot (c_1\vec{u} + c_2\vec{v}) = c_1 \nabla f(\vec{a}) \cdot \vec{u} + c_2 \nabla f(\vec{a}) \cdot \vec{v}$$

$$= c_1 D_{\vec{u}} f(\vec{a}) + c_2 D_{\vec{v}} f(\vec{a}) = 0.$$

(d) (5 points) Let (a, b, c) be a point in $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^3 + e^y + xz^3$. Then the following matrix is invertible.

$$\begin{bmatrix} f_{xx}(a,b,c) & f_{xy}(a,b,c) & f_{xz}(a,b,c) \\ f_{yx}(a,b,c) & f_{yy}(a,b,c) & f_{yz}(a,b,c) \\ f_{zx}(a,b,c) & f_{zy}(a,b,c) & f_{zz}(a,b,c) \end{bmatrix}$$

Answer: SOMETIMES

$f_x(x,y,z) = 3x^2 + z^3$ $\quad f_y(x,y,z) = e^y$ $\quad f_z(x,y,z) = 3xz^2$

$f_{xx}(x,y,z) = 6x$

$f_{xy}(x,y,z) = 0$ $\quad f_{yy}(x,y,z) = e^y$

$f_{xz}(x,y,z) = 3z^2$ $\quad f_{yz}(x,y,z) = 0$ $\quad f_{zz}(x,y,z) = 6xz.$

By Clairaut's, $A = \begin{bmatrix} 6a & 0 & 3c^2 \\ 0 & e^b & 0 \\ 3c^2 & 0 & 6ac \end{bmatrix}$. If $(a,b,c) = (0,0,0)$, then

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ isn't invertible. If $(a,b,c) = (0,0,1)$, then $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ is invertible

3. Suppose the depth of a pond at the point (x, y) is given by the function $f(x, y) = 50 - 4x^2y^2 - x^2 - 2y^2$. A child is standing at the point $(-3, 1)$ and can only just touch the bottom.

(a) (6 points) In what direction could the child walk so that the depth decreases most rapidly (i.e., so they can easily touch the bottom)? Your answer does *not* need to be expressed as a unit vector

$$\nabla f(x,y) = \begin{bmatrix} -8xy^2 - 2x \\ -8x^2y - 4y \end{bmatrix} \Rightarrow \nabla f(-3,1) = \begin{bmatrix} -8\cdot -3 \cdot 1^2 - 2(-3) \\ -8(-3)^2 \cdot 1 - 4 \cdot 1 \end{bmatrix} = \boxed{\begin{bmatrix} 30 \\ -76 \end{bmatrix}}$$

If child moves in the direction of $-\nabla f(-3,1) = \begin{bmatrix} -30 \\ 76 \end{bmatrix}$, the depth will decrease most rapidly.

(b) (4 points) In what direction could the child walk so that the depth remains constant? Justify your answer. Your answer does *not* need to be expressed as a unit vector.

Child should move perpendicular to the gradient $\nabla f(-3,1)$ to stay on the same level curve : $\vec{v} = \begin{bmatrix} 76 \\ 30 \end{bmatrix}$ will work.

(c) (8 points) If the child swims directly toward the center of the pond (i.e., toward $(0, 0)$), at what rate is the depth changing at the moment when they start swimming?

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{bmatrix} \quad D_{\vec{v}} f(-3,1) = \nabla f(-3,1) \cdot \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ -76 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{bmatrix}$$

$$= \frac{90}{\sqrt{10}} + \frac{76}{\sqrt{10}} = \frac{166}{\sqrt{10}}$$

4. (10 points) Let $f(x,y) = xye^y$. Find all points (a,b) such that the tangent plane to the graph of f at $(a,b,f(a,b))$ is parallel to the plane $8y - 2z = 10$.

normal to tangent plane is $\begin{bmatrix} f_x(a,b) \\ f_y(a,b) \\ -1 \end{bmatrix}$.

$f_x(x,y) = ye^y$

$f_y(x,y) = xe^y + xye^y$.

$\Rightarrow \vec{n} = \begin{bmatrix} be^b \\ ae^b + abe^b \\ -1 \end{bmatrix}$ must be parallel to $\begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$

To make the z-components equal, we scale by $\frac{1}{2}$ and get $\begin{bmatrix} be^b \\ ae^b + abe^b \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$

$\Rightarrow be^b = 0 \Rightarrow b = 0$

$\Rightarrow ae^b + abe^b = 4 \xrightarrow{b=0} a = 4.$

Thus the only point is $(4,0)$.

5. (16 points) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (x^2 + z, e^x)$, and let $x(r, \theta, z), y(r, \theta, z)$ and $z(r, \theta, z)$ be the expressions for the Cartesian coordinates (x, y, z) in terms of the cylindrical coordinates (r, θ, z). Find the derivative (i.e. Jacobian matrix) $Dg(r, \theta, z)$ of the function $g(r, \theta, z) = f(x(r, \theta, z), y(r, \theta, z), z(r, \theta, z))$ at the point with Cartesian coordinates $(x, y, z) = (0, 1, 2)$.

$(x,y,z) = (0,1,2)$ in cylindrical coordinates is $(r,\theta,z) = (1, \frac{\pi}{2}, 2)$.

$$Dg(1, \tfrac{\pi}{2}, 2) = Df(0,1,2)\, D\bar{x}(1, \tfrac{\pi}{2}, 2)$$

$$Df(x,y,z) = \begin{bmatrix} 2x & 0 & 1 \\ e^x & 0 & 0 \end{bmatrix} \Rightarrow Df(0,1,2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\bar{x}(r,\theta,z) = \begin{bmatrix} x(r,\theta,z) \\ y(r,\theta,z) \\ z(r,\theta,z) \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

$$D\bar{x}(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow D\bar{x}(1, \tfrac{\pi}{2}, 2) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow Dg(1, \tfrac{\pi}{2}, 2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

6. Define planes P_1, P_2 and P_3 as follows:

$$\begin{cases} P_1: & x & & - & z & = & 0 \\ P_2: & & y & + & 2z & = & 1 \\ P_3: & x & + \; y & + & z & = & -2 \end{cases}$$

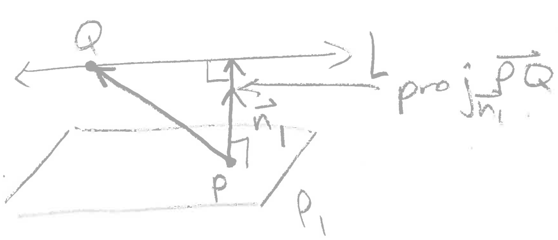
(a) (8 points) Verify that the line of intersection of the planes P_2 and P_3 does not intersect the plane P_1.

Line of intersection has direction $\vec{v} = \vec{n_2} \times \vec{n_3}$, where $\vec{n_2}$ is the normal to P_2 and $\vec{n_3}$ is the normal to P_3. $\Rightarrow \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k}$

The line won't intersect P_1 if $\vec{v}$ is parallel to P_1 (ie if $\vec{v} \perp \vec{n_1}$). $\vec{v} \cdot \vec{n_1} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$

Thus the line doesn't intersect P_1.

(b) (8 points) Find the distance from P_1 to the line of intersection of P_2 and P_3.



point P on P_1 is $(0,0,0)$
point Q on L is $(-3,1,0)$
(observed by letting $z=0$).

$\Rightarrow \vec{PQ} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

distance $= \|\text{proj}_{\vec{n_1}} \vec{PQ}\| = \left\| \dfrac{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \left\| \dfrac{-3}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \dfrac{3}{2}\sqrt{2} = \dfrac{3}{\sqrt{2}}$

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