Northwestern University **Name:** Solutions

Math 290-3: Midterm 1

Spring Quarter 2015

Thursday, April 30, 2015

Put a check mark next to your section:

Davis (10am)		Canez	
Peterson		Davis (12pm)	

Question	Possible points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 9 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

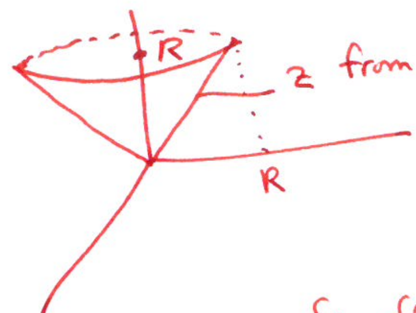
1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) The following integrals are equal:

$$\int_0^{2\pi}\int_0^{R}\int_r^{R} \theta\, r\, dz\, dr\, d\theta \quad \text{and} \quad \int_0^{2\pi}\int_0^{\pi/4}\int_0^{R/\cos\phi} \theta\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

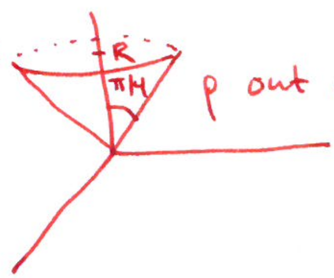
Answer: TRUE

region for 1st integral:



z from cone $z=r$ up to plane $z=R$

for 2nd:



ρ out as far as plane $z=R$ which is $\rho = R/\cos\phi$

So same region and same integrand θ

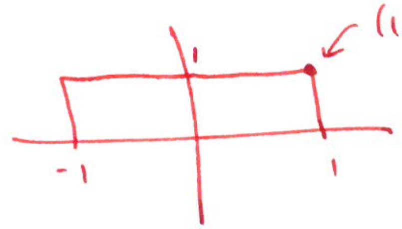
   (b) Suppose that R is a rectangle in $\mathbb{R}^2$ and that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. If $\iint_R f(x,y)\, dA = 0$, then every Riemann sum of f has the value zero.

Answer: FALSE

$f(x,y) = x \quad R = [-1,1] \times [0,1]$

$\iint_R f(x,y)\, dA = 0$ by symmetry but

for partition with entire rectangle



(1,1) only sample point

have Riemann Sum $= f(1,1) \cdot \text{area}$
$= 1 \cdot 2 = 2 \neq 0$

(c) The following equality holds:

$$\int_0^4 \int_{-1}^1 \int_0^2 (4 - 4y + y^{101} e^{\cos(y^2 z)+\sin x + z})\, dz\, dy\, dx = \frac{8}{3}$$

Answer: FALSE

$$-4y + y^{101} e^{\cos(y^2 z) + \sin x + z} \quad \text{odd}$$

changes sign, changes sign, keeps sign — with respect to y under $y \to -y$

So $\iiint (-4y + y^{101} e^{\cos(y^2 z) + \sin x + z})\, dV = 0$ by symmetry.

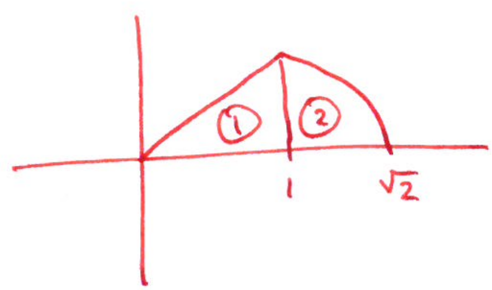
Left with $\int_0^4 \int_{-1}^1 \int_0^2 4 \, dz\, dy\, dx = 4 \cdot 4 \cdot 2 \cdot 2 = 64 \neq 8/3$

(d) The following equality holds:

$$\int_0^1 \int_0^{x} dy\, dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} dy\, dx = \frac{\pi}{4}$$

Answer: TRUE

regions are



So combined region

is $\frac{1}{8}$th of disk of radius $\sqrt{2}$

$$\int_0^1 \int_0^x dy\, dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} dy\, dx = \iint dA = \text{area}$$

$$= \frac{1}{8}\left(\pi r^2 \text{ with } r = \sqrt{2}\right) = \pi/4$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

(a) For a number k,

$$\int_{10}^{20}\int_{0}^{e^x} [x^2 + y^2 + (k^2+1)^3]\, dy\, dx \geq 0.$$

Answer: Always

Since $x^2+y^2 \geq 0$ for all (x,y), and $(k^2+1)^3 \geq 1 > 0$ for every k, we have

$x^2+y^2+(k^2+1)^3 \geq 1 > 0$ for every x, y, and k.

Therefore, by monotonicity,

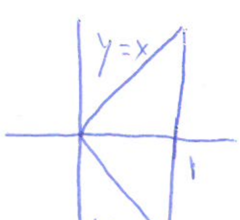
$$\int_{10}^{20}\int_{0}^{e^x} [x^2+y^2+(k^2+1)^3]\, dy dx \geq \int_{10}^{20}\int_{0}^{e^x} 0 \, dy\, dx = 0,$$

so the statement is always true.

(b) For a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\int_0^1 \int_{-x}^{x} f(x,y)dydx = \int_{-1}^{1}\int_0^1 vf(v,uv)dvdu.$$

Answer: Always



The region of integration on the left is the triangle bounded by $x=1$, $y=x$, and $y=-x$.

Defining $x=v$ and $y=uv$ (so $u=\frac{y}{x}$), this region can be expressed in terms of u and v as $0 \leq v \leq 1$, $-1 \leq u \leq 1$.

Now, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} 0 & 1 \\ v & u\end{pmatrix}\right| = |-v| = v$, so that

$$\int_0^1\int_{-x}^{x} f(x,y)dydx = \int_{-1}^{1}\int_0^1 f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu = \int_{-1}^{1}\int_0^1 f(v,uv)v\,dvdu.$$

Therefore, the statement is always true.

(c) For $k > 0$, if E is the solid consisting of all points satisfying the inequalities $k \leq \sqrt{x^2 + y^2 + z^2} \leq 2k$, then

$$\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\, dV = 4\pi^2 \ln(2).$$

Answer: Never

In spherical coordinates, E is given by $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$, $k \leq \rho \leq 2k$.
Thus,

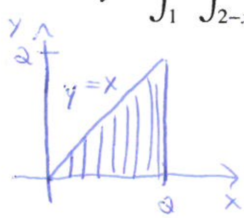
$$\iiint_E \frac{1}{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_k^{2k} \frac{1}{\rho^3}\rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_k^{2k} \rho^{-1}\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \sin(\varphi)\underbrace{\left[\ln(2k) - \ln(k)\right]}_{=\ln\left(\frac{2k}{k}\right) = \ln(2)} d\varphi\, d\theta$$
$$= \int_0^{2\pi} 2\ln(2)\, d\theta = 4\pi \ln(2) \neq 4\pi^2 \ln(2)$$

(d) For a number k,

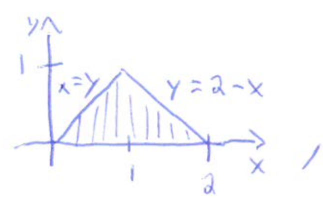
$$\int_0^2 \int_0^x k e^{kx^{10}}\, dy\, dx = \int_0^1 \int_y^{2-y} e^{kx^{10}}\, dx\, dy + \int_1^2 \int_{2-x}^{x} e^{kx^{10}}\, dy\, dx.$$

Answer: Sometimes

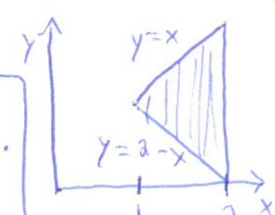
The region on the left is



The region of the first integral on the right is



while that of the second region is



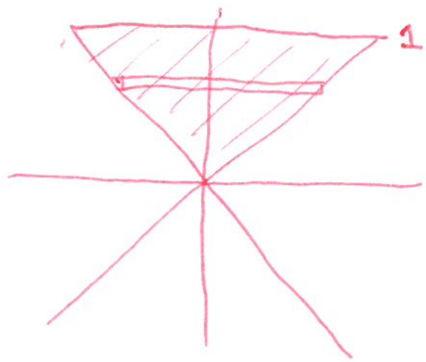
Therefore the RHS is equal to $\int_0^2\int_0^x e^{kx^{10}}\, dy\, dx$.
In other words,

$$k\int_0^2\int_0^x e^{kx^{10}}\, dy\, dx = \int_0^2\int_0^x e^{kx^{10}}\, dy\, dx.$$

If $k=0$, this says $0 = \int_0^2\int_0^x 1\, dy\, dx = 2$, making the statement false.
If $k=1$, this says $\int_0^2\int_0^x e^{x^{10}}\, dy\, dx = \int_0^2\int_0^x e^{x^{10}}\, dy\, dx$, making the statement true.

3. Determine the value of the following expression.

$$\int_{-1}^{0}\int_{-x}^{1} e^{y^2}\,dy\,dx + \int_{0}^{1}\int_{x}^{1} e^{y^2}\,dy\,dx$$



y goes from 0 to 1.

x goes from $x=-y$ to $x=y$

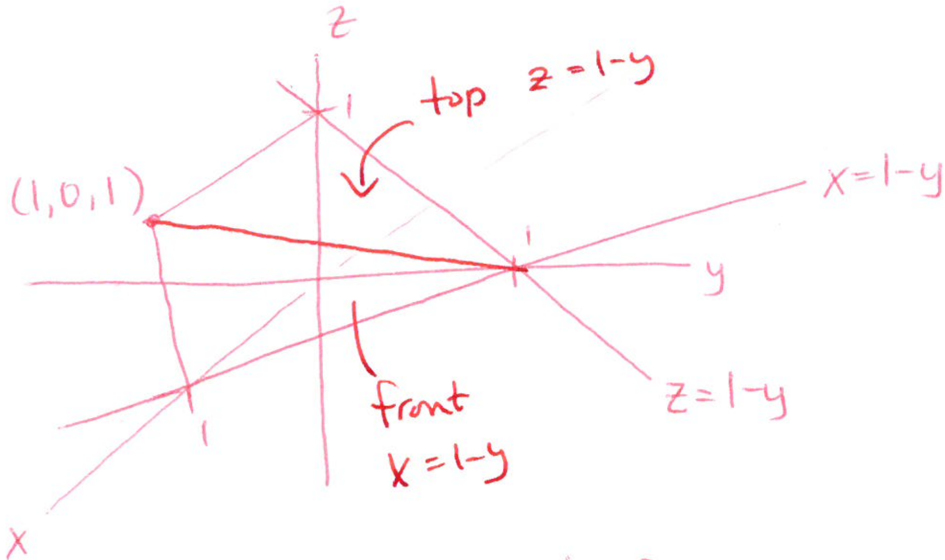
$$\int_0^1\int_{-y}^{y} e^{y^2}\,dx\,dy = \int_0^1 2e^{y^2}y\,dy$$

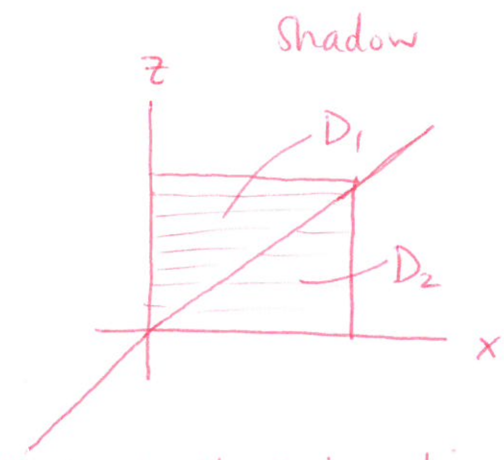
$$= e^{y^2}\Big]_0^1$$

$$= e-1$$

4. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Rewrite the following triple integral with respect to the order $dy\,dx\,dz$.

$$\int_0^1 \int_0^{1-y} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy$$





In D_1, $0 \le y \le 1-z$.

In D_2, $0 \le y \le 1-x$.

$x=z$ is the intersection of the two planes $x=1-y$, $z=1-y$

So,

$$\iint_{D_1} \int_0^{1-z} f(x,y,z)\,dy\,dx\,dz + \iint_{D_2} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

$$= \boxed{\int_0^1 \int_0^z \int_0^{1-z} f(x,y,z)\,dy\,dx\,dz + \int_0^1 \int_z^1 \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz}$$

5. Find a change of variables $u = u(x, y)$ and $v = v(x, y)$ under which the double integral $\iint_D (6y + 3)\, dA$, where D is the region in the xy-plane bounded by the curves

$$y = -x,\ y = 1 - x,\ x = y^2, \text{ and } x = y^2 - 1,$$ and above the x-axis $(y \geq 0)$

becomes the double integral of a constant over a rectangle in the uv-plane. Setup the integral in terms of u and v, but do NOT evaluate it.

$x+y=0$

$x+y=1$

$y^2-x=0$

$y^2-x=1$

$u = x+y$, $0 \le u \le 1$

$v = y^2 - x$, $0 \le v \le 1$

$$\left|\frac{1}{\det\begin{pmatrix}1 & 1\\ -1 & 2y\end{pmatrix}}\right| = \left|\frac{1}{2y+1}\right| = \frac{1}{2y+1}$$ since $y \geq 0$ in the region.

$$\iint_D (6y+3)\, dA = \int_0^1\int_0^1 6y+3 \cdot \frac{1}{2y+1}\, du\, dv = \int_0^1\int_0^1 3\, du\, dv.$$

(5) $u = x+y$
$v = x-y^2$ or y^2-x

(2) det $= 2y+1$ (1) $\|\det\|$

(2) $\frac{1}{\det}$

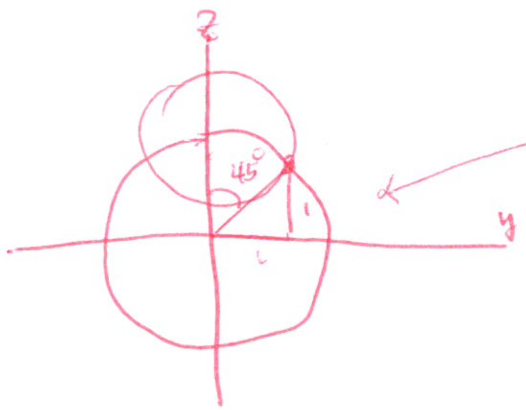
$\int_0^1\int_0^1$ or $\int_{-1}^0\int_0^1 3\, du\, dv.$

(2) (3)

6. Determine the value of

$$\iiint_E \frac{1}{x^2+y^2+z^2}\,dV$$

where E is the region above the xy-plane which lies outside the sphere $x^2+y^2+z^2=2$ and inside the sphere $x^2+y^2+z^2=2z$.



$\rho^2 = 2 \Rightarrow \rho = \sqrt{2}$

$\rho^2 = 2\rho\cos\phi \rightarrow \rho = 2\cos\phi$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\sqrt{2}}^{2\cos\phi} \frac{\rho^2\sin\phi}{\rho^2}\,d\rho\,d\phi\,d\theta$$

$= 2\pi \int_0^{\frac{\pi}{4}} (2\cos\phi - \sqrt{2})\sin\phi\,d\phi$

$= 2\pi \int_0^{\frac{\pi}{4}} (2\sin\phi\cos\phi - \sqrt{2}\sin\phi)\,d\phi$

$= 2\pi \left(\sin^2\phi + \sqrt{2}\cos\phi\right]_0^{\frac{\pi}{4}}$

$= 2\pi \left(\left(\frac{\sqrt{2}}{2}\right)^2 + \frac{\sqrt{2}\sqrt{2}}{2} - 0 - \sqrt{2}\right)$

$= 2\pi\left(\frac{1}{2} + 1 - \sqrt{2}\right)$

$= 2\pi\left(\frac{3}{2} - \sqrt{2}\right)$