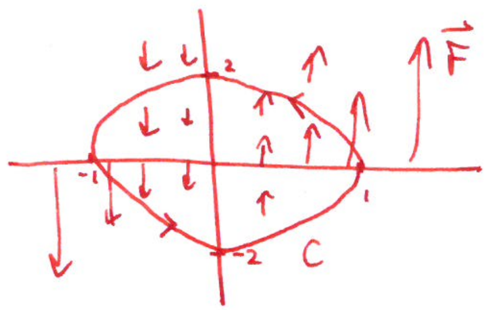
1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

(a) Let $\mathbf{F}(x, y) = (0, x)$ and let $C$ be the ellipse $x^2 + y^2/4 = 1$ oriented counter-clockwise. Then $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$.

Answer: TRUE




angle between $\vec{F}$ and tangent vector always $\leq \pi/2$ so $\vec{F} \cdot$ tangent $> 0$ always and so integral is $> 0$

(b) Let $\mathbf{F}(x, y) = (2xy + \sin y, x^2 + x\cos y)$ and let $C$ be the curve with parametric equations $\mathbf{x}(t) = (\sin(\pi t), t^2)$ for $-1 \leq t \leq 1$. Then $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$.

Answer: FALSE

$\vec{F} = \nabla(x^2y + x\sin y)$ is conservative

and $C$ is closed since

$\vec{x}(-1) = (0, 1) = \vec{x}(1)$

So line integral is $0$

(c) Let $D$ be the region in $\mathbb{R}^2$ obtained by removing the origin. If $\mathbf{F}$ is a $C^1$ vector field on $D$ such that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ at each point of $D$, then $\oint_C \mathbf{F}\cdot d\mathbf{s} = 0$ for every closed curve $C$ in $D$.

Answer: FALSE

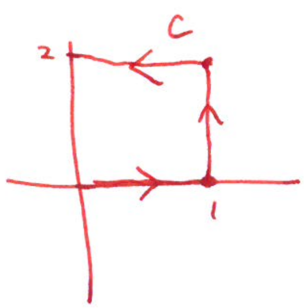
Let $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$. Then $\operatorname{curl}\vec{F} = \vec{0}$ on all of $D$

but $\displaystyle\int_{\substack{\text{unit circle,}\\ \text{counterclockwise}}} \vec{F}\cdot d\vec{s} = 2\pi$ is not zero

(d) Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$, let $D$ be the rectangle $[0, 1] \times [0, 2]$ and let $C$ be the curve consisting of the line segment from $(0, 0)$ to $(1, 0)$, followed by the line segment from $(1, 0)$ to $(1, 2)$, followed by the line segment from $(1, 2)$ to $(0, 2)$. Assume that $\mathbf{F}$ has continuous partial derivatives in $D$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_0^1\int_0^2 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx + \int_0^2 Q(0, t)\,dt.$$

Answer: TRUE




Let $C_1$ be segment from $(0,2)$ to $(0,0)$.

Then

$$\int_C \vec{F}\cdot d\vec{s} = \underbrace{\int_{C+C_1} \vec{F}\cdot d\vec{s}}_{\int_0^1\int_0^2 (Q_x - P_y)\,dy\,dx \text{ by Green's}} - \underbrace{\int_{C_1} \vec{F}\cdot d\vec{s}}$$

$\vec{x}(t) = (0,t)$, $0 \le t \le 2$

equations for $C_1$ opposite

so $-\displaystyle\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{C_1 \text{opp}} \vec{F}\cdot d\vec{s} = \int_0^2 Q(0,t)\,dt.$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

   (a) For numbers $a < b$, let $\mathbf{x}(t) = (x(t), y(t))$ with $a \le t \le b$ be parametric equations for a smooth curve such that $\mathbf{x}(t)$ lies on the unit circle for all $t$. Then

$$\int_a^b \|\mathbf{x}'(t)\|\, dt = 2\pi.$$

Answer: Sometimes

Counter-Example: $\vec{x}(t) = (\cos(t), \sin(t)),\ 0 \le t \le \frac{\pi}{2}$.

Then $\int_0^{\pi/2} \|\vec{x}'(t)\| dt = \int_0^{\pi/2} 1\, dt = \frac{\pi}{2}$. (Statement is false.)

Example: $\vec{x}(t) = (\cos(t), \sin(t)),\ 0 \le t \le 2\pi$.

Then $\int_0^{2\pi} \|\vec{x}'(t)\| dt = \int_0^{2\pi} 1\, dt = 2\pi$. (Statement is true.)

   (b) Let $C$ be the portion of the curve defined by $1 = ye^{-x^2}$ which starts at $(-1, e)$ and ends at $(1, e)$. For a real number $k$,

$$\int_C -2xye^{-x^2} dx + (e^{-x^2} + k^2) dy > 0.$$

Answer: Never

Note that $(-2xye^{-x^2})\vec{i} + (e^{-x^2} + k^2)\vec{j} = \nabla(ye^{-x^2} + k^2 y)$, so by the Fundamental Theorem for Line Integrals we have

$$\int_C (-2xye^{-x^2})dx + (e^{-x^2} + k^2)\, dy = \int_C \nabla(ye^{-x^2} + k^2 y) \cdot d\vec{s}$$

$$= ye^{-x^2} + k^2 y \Big|_{(-1,e)}^{(1,e)} = e e^{-(1)^2} + k^2 e - (e e^{-(-1)^2} + k^2 e) = 0,$$

for every $k$. So, the statement is never true.

(c) For a $C^1$ vector field $\mathbf{F}(x, y)$ on $\mathbb{R}^2$ such that $\operatorname{div}(\mathbf{F}) = 0$ everywhere, $(\operatorname{curl}\mathbf{F})(p) \neq \mathbf{0}$ for every point $p$ in $\mathbb{R}^2$.

Answer: Sometimes

Example: $\vec{F} = \begin{pmatrix} y \\ -x \end{pmatrix}$. Then $\operatorname{div}\vec{F} = 0$, but $\operatorname{curl}\vec{F} = \begin{pmatrix} 0-0 \\ 0-0 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \neq \vec{0}$ at every point.

Counterexample: $\vec{F} = \vec{0}$. Then $\operatorname{div}\vec{F} = 0$ everywhere, and $\operatorname{curl}\vec{F} = \vec{0}$ everywhere as well.

(d) For a **nonzero** $C^2$ vector field $\mathbf{F}$ on $\mathbb{R}^3$, we have $\operatorname{curl}\mathbf{F} = \nabla(\operatorname{div}\mathbf{F})$.

Answer: Sometimes

Example: $\vec{F} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then $\operatorname{curl}\vec{F} = \vec{0} = \nabla(0) = \nabla(\operatorname{div}\vec{F})$.

Counterexample: $\vec{F} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$. Then $\operatorname{curl}\vec{F} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, but $\operatorname{div}\vec{F} = 0$, so $\nabla(\operatorname{div}\vec{F}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

3. Determine the value of the scalar line integral

$$\int_C (2xy - yz)\, ds$$

where $C$ is the intersection of the cylinder $y^2 + z^2 = 1$ and the plane $z = x$.

Method 1: Parametrize the curve & integrate.

$x = z = \cos t, \quad y = \sin t$

$$\int_0^{2\pi} (2\cos t \cdot \sin t - \sin t \cdot \cos t) \sqrt{(-\sin t)^2 + \cos^2 t + (-\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \cos t \cdot \sin t \sqrt{1 + \sin^2 t}\, dt \qquad \begin{cases} u = 1 + \sin^2 t \\ du = 2 \sin t \cos t\, dt \end{cases}$$

$$= \frac{1}{2} \int_{t=0}^{t=2\pi} u^{1/2}\, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_{t=0}^{t=2\pi} = 0.$$

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Method 2: $C$ is symmetric wrt the $xz$-plane and the function $2xy - yz$ is odd as a function of $y$, so the integral is $0$.

4. Compute the vector line integral

$$\int_C (y + \sin y + e^{x^4})\,dx + (y + (x-1)\cos y)\,dy$$

where $C$ is the left half of the circle $(x-1)^2 + y^2 = 1$ oriented clockwise.


Set $P(x,y) = y + \sin y + e^{x^4}$, $Q(x,y) = y + (x-1)\cos y$, $D$ region bounded by $\tilde{C} - C$.

$$\int_{\tilde{C}-C} P(x,y)\,dx + Q(x,y)\,dy$$

$$\overset{\text{Green's}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \cos y - 1 - \cos y \, dA$$

$$= -\iint_D dA = -\frac{\pi}{2}.$$

So, $\displaystyle\int_{\tilde{C}} P\,dx + Q\,dy - \int_C P\,dx + Q\,dy = -\frac{\pi}{2}$

$\displaystyle\Rightarrow \int_C P\,dx + Q\,dy = \int_{\tilde{C}} P\,dx + Q\,dy + \frac{\pi}{2}.$

Parametrize $\tilde{C}$ by $x(t) = (1, t)$, $t$ from $-1$ to $1$. Then $x'(t) = (0,1)$

$$\int_{\tilde{C}} P\,dx + Q\,dy$$

$$= \int_{-1}^{1} \left[P(x(t))\cdot 0 + \left(t + (1-1)\cos(t)\right)\cdot 1\right] dt$$

$$= \int_{-1}^{1} t\,dt = \frac{t^2}{2}\Big]_{-1}^{1} = 0.$$

So $\displaystyle\int_C (y + \sin y + e^{x^4})\,dx + (y + (x-1)\cos y)\,dy = \frac{\pi}{2}.$

5. (This question has **two** parts.) Let **F** be the vector field on $\mathbb{R}^2$ defined by

$$\mathbf{F}(x,y) = (\underbrace{ye^y + y^2 - y\pi\sin(xy\pi)}_{P})\mathbf{i} + (\underbrace{xe^y + xye^y + 2yx - x\pi\sin(xy\pi)}_{Q})\mathbf{j}.$$

(a) Show that **F** is conservative on $\mathbb{R}^2$.

want $\nabla f = F$.

$\Leftrightarrow$ $f_x = P$, $f_y = Q$

So $f = \int P\,dx = xye^y + xy^2 + \cos(xy\pi) + g(y)$

$\Rightarrow$ $f_y = xe^y + xye^y + 2xy - x\pi\sin(xy\pi) + g_y(y) = Q(x,y)$

$\Rightarrow$ $g_y(y) = 0$

$\Rightarrow$ $g(y) = C$

Can pick it to be 0.

So $f(x,y) = xye^y + xy^2 + \cos(xy\pi)$

has $\nabla f = F$.

$\Rightarrow$ $F$ conservative.

(b) Find the value of the vector line integral

$$\int_C (y + ye^y + y^2 - y\pi\sin(xy\pi))\,dx + (-x + xe^y + xye^y + 2yx - x\pi\sin(xy\pi))\,dy$$

where $C$ is the piece of the parabola $x = y^2 - 1$ which starts at $(0, -1)$ and ends at $(0, 1)$.

Solution

This line integral is

$$\int_C P\,dx + Q\,dy + \int_C y\,dx - x\,dy$$

where $P\vec{i} + Q\vec{j} = F$ from part a).

$$= f(0,1) - f(0,-1) + \int_C y\,dx - x\,dy$$

$$= \cos(0) - \cos(0) + \int_C y\,dx - x\,dy$$

$$= \int_C y\,dx - x\,dy$$

Parametrize $C$:

$x(t) = (t^2 - 1, t)$, $t$ from $-1$ to $1$

$x'(t) = (2t, 1)$

$$\int_C y\,dx - x\,dy = \int_{-1}^{1} (2t^2 - t^2 + 1)\,dt$$

$$= \int_{-1}^{1} (t^2 + 1)\,dt$$

$$= \left.\frac{t^3}{3} + t\right]_{-1}^{1}$$

$$= \frac{1}{3} + 1 + \frac{1}{3} + 1$$

$$= 2 + \frac{2}{3}$$

$$= \frac{8}{3}.$$

Result: $\frac{8}{3}$.

6. Compute the vector line integral

$$\int_C (z + y\sin^2(z+1))\,dx - x\cos^2(z+1)\,dy + z^{100}e^{x\cos y}\,dz$$

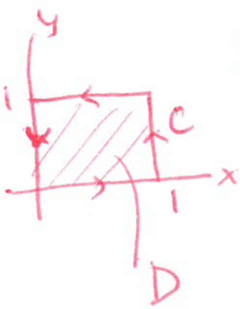
where $C$ is the square $[0,1]\times[0,1]$ in the $xy$-plane oriented counterclockwise when viewed from the positive $z$-direction.

$z=0$ (xy plane)

$\Rightarrow$ Integral $= \int_C y\cdot\sin^2(1)\,dx - x\cdot\cos^2(1)\,dy$

$= \iint_D -\cos^2(1) - \sin^2(1)\,dA$

Green's thm




$= \iint_D -dA$

$= -(\text{area of } D)$

$= -1$