Math 291-1: Final Exam Solutions
Northwestern University, Fall 2020

1. Determine whether each of the following statements is true or false, and provide justification for your answer.

(a) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^3$ are linearly independent over $\mathbb{C}$, then they are linearly independent over $\mathbb{R}$.

(b) If A, B, C are 2×2 matrices such that $ABC = 0$, then none of A, B, C are invertible.

(c) If $T : P_2(\mathbb{R}) \to \mathbb{R}^+$ is linear, where $\mathbb{R}^+$ is the vector space of positive real numbers from Problem 7 of Homework 6, then $T(0) = 1$.

Solution. (a) This is true. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^3$ to be linearly independent over $\mathbb{C}$ means that the only complex scalars $a, b, c \in \mathbb{C}$ satisfying

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

are $a = b = c = 0$. But then in particular, the only *real* scalars satisfying this equation are also $a = b = c = 0$, since the real scalars satisfying this are among the complex scalars satisfying it. So $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are also linearly independent over $\mathbb{R}$.

(b) This is false. Take $B = I$ for instance, and $A = C = 0$. Then $ABC = 0$ and $B = I$ is invertible

(c) This is true. Any linear transformation sends zero vector to zero vector, and the zero vector of $\mathbb{R}^+$ is indeed 1, so $T(0) = 1$ holds. □

2. Suppose A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Show that $A\mathbf{x} = \mathbf{b}$ has no solution if and only if $\operatorname{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \operatorname{rank} A + 1$, where $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is the $m \times (n+1)$ matrix whose first n columns are those of A and whose final column is $\mathbf{b}$.

Proof. The equation $A\mathbf{x} = \mathbf{b}$ has no solution if and only if the reduced row-echelon form of the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ has a row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

since this corresponds to the impossible equation $0 = 1$. But $\operatorname{rref} \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ has such a row if and only if the final column gives rise to one more pivot not present in the portion corresponding to A alone. Since the number of pivots in the portion of $\operatorname{rref} \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ corresponding to A has as many pivots as $\operatorname{rref}(A)$, this final column gives rise to one more pivot if and only if the number of pivots in $\operatorname{rref} \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is exactly one more than the number of pivots in $\operatorname{rref}(A)$, which is equivalent to $\operatorname{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \operatorname{rank} A + 1$ as stated. □

3. Suppose P is a 2×2 matrix of rank 1 such that $P^2 = P$ with the property that anything in $\operatorname{im} P$ is perpendicular to anything in $\ker P$. Show that P is the matrix of an orthogonal projection. (Hint: First determine the line onto which P should orthogonally project an arbitrary vector, and for this think about the effect which P has on something in $\operatorname{im} P$.)

Proof. Let L be the image of P, which, being of dimension 1, is a line through the origin in $\mathbb{R}^2$. We claim that P is the matrix which describes orthogonal projection onto this line. Let Q be the matrix of orthogonal projection onto this line, so we must show that $P = Q$.

Take a nonzero vector $\mathbf{u} \in L$ and a nonzero vector $\mathbf{v} \in \ker P$. Then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent since they are perpendicular, so they automatically span $\mathbb{R}^2$. Thus if $\mathbf{x} \in \mathbb{R}^2$ we have

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v}$$

for some $a, b \in \mathbb{R}$. Then:

$$P\mathbf{x} = aP\mathbf{u} + bP\mathbf{v} = aP\mathbf{u},$$

where $P\mathbf{v} = \mathbf{0}$ since $\mathbf{v} \in \ker P$. Now, since $\mathbf{u}$ is in $L = \operatorname{im} P$, we have $\mathbf{u} = P\mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^2$. Thus:

$$P\mathbf{u} = P(P\mathbf{w}) = P^2\mathbf{w} = P\mathbf{w} = \mathbf{u}$$

and hence $P\mathbf{x} = a\mathbf{u}$. On the other hand, $Q\mathbf{u} = \mathbf{u}$ since orthogonal projection of $\mathbf{u} \in L$ onto L leaves $\mathbf{u}$ unchanged, and $Q\mathbf{v} = \mathbf{0}$ since vectors perpendicular to L orthogonally project to $\mathbf{0}$, so

$$Q\mathbf{x} = aQ\mathbf{u} + bQ\mathbf{v} = a\mathbf{u}$$

as well. Hence $P\mathbf{x} = Q\mathbf{x}$ for all $x \in \mathbb{R}^2$, which means that $P = Q$ as desired. (The overarching point is that, due to linearity, the behavior of P on a basis determines its behavior on everything.) $\square$

4. Suppose A and B are $n \times n$ matrices, and that A is row-equivalent to I. Show that AB is row-equivalent to B. (Careful: it is NOT true in general that $\operatorname{rref}(CD) = \operatorname{rref}(C)\operatorname{rref}(D)$.)

Proof 1. Since A is row-equivalent to I, there exist elementary matrices (i.e. the matrix form of elementary row operations) which transform A into I:

$$E_m \cdots E_1 A = I.$$

Performing these same row operations to AB is the same as multiplying by the same elementary matrices, which gives:

$$E_m \cdots E_1(AB) = (E_m \cdots E_1 A)B = IB = B.$$

Thus AB is row-equivalent to B. $\square$

Proof 2. To show that AB is row-equivalent to B it is equivalent to show that $(AB)\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions. (This was on Homework 3.) If $\mathbf{x} \in \mathbb{R}^n$ satisfies $B\mathbf{x} = \mathbf{0}$, then it also satisfies $(AB)\mathbf{x} = \mathbf{0}$ since:

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Conversely, if $\mathbf{x}$ satisfies $(AB)\mathbf{x} = \mathbf{0}$, then multiplying this equality by A^{-1} (which exists since $\operatorname{rref}(A) = I$) on the left of both sides gives

$$B\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{x}$ satisfies $(AB)\mathbf{x} = \mathbf{0}$ if and only if it satisfies $B\mathbf{x} = \mathbf{0}$, which implies that AB and B are row-equivalent. $\square$

5. Suppose U is a subspace of $P_n(\mathbb{R})$ with the property that whenever $p(x)$ is in U, we have that its derivative $p'(x)$ is also in U. If $x^n \in U$, show that $U = P_n(\mathbb{R})$.

Proof. If $x^n \in U$, then by the property of U given in the setup, we have that its derivative nx^{n-1} is also in U. But then applying this same property implies that the derivative of *this* is also in U: $n(n-1)x^{n-2} \in U$. And so on, continuing to take more and more derivatives gives

$$n(n-1)(n-2)x^{n-3} \in U, \ \ldots, n(n-1)(n-2)\cdots 3 \cdot 2x \in U, \ n! \in U.$$

Since U is closed under scalar multiplication, dividing each of these elements of U by the appropriate scalar gives

$$x^{n-1} \in U, \ x^{n-2} \in U, \ \ldots, \ x \in U, \ 1 \in U.$$

Thus U contains all of the basis vectors $1, x, x^2, \ldots, x^n$ of $P_n(\mathbb{R})$, which means that $U = P_n(\mathbb{R})$. $\square$

6. Suppose V is a 3-dimensional vector space and that $T : V \to V$ is a linear transformation such that $T^5 = 0$. Show that $T^3 = 0$. (You cannot just quote a remark you might have seen somewhere which says that this is true—you must prove it. You can, however, look around at old exam problems to get an idea for what to do. Note the exponent in T^5 is 5, not 4.)

Proof. Aiming for a contradiction, suppose $T^3 \neq 0$. Then there exists a nonzero $v \in V$ such that $T^3 v \neq 0$. Consider the vectors v, Tv, T^2v, T^3v, T^4v and suppose $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$ satisfy

$$a_1 v + a_2 Tv + a_3 T^2 v + a_4 T^3 v + a_5 T^4 v = 0.$$

Applying T to both sides gives

$$a_1 Tv + a_2 T^2 v + a_3 T^3 v + a_4 T^4 v = 0,$$

where use the fact that $T^5 = 0$ in order to say that $a_5 T^5 v = 0$. Applying T again gives:

$$a_1 T^2 v + a_2 T^3 v + a_3 T^4 v = 0,$$

and again gives

$$a_1 T^3 v + a_2 T^4 v = 0,$$

and one more gives

$$a_1 T^4 v = 0.$$

Now, there are two possibilities: either $T^4 v \neq 0$ or $T^4 v = 0$. If $T^4 v \neq 0$, then we must have $a_1 = 0$. But then the previous equation becomes $a_2 T^4 v = 0$, so that again since $T^4 v \neq 0$ we must have $a_2 = 0$. Then the equation before this becomes $a_3 T^4 v = 0$, so $a_3 = 0$; the equation before this becomes $a_4 T^4 v = 0$, so $a_4 = 0$; and finally our original equation is simply $a_5 T^4 v = 0$, so that $a_5 = 0$. Thus in this case the only coefficients satisfying

$$a_1 v + a_2 Tv + a_3 T^2 v + a_4 T^3 v + a_5 T^4 v = 0$$

are $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, so v, Tv, T^2v, T^3v, T^4v are linearly independent, which is not possible since $\dim V = 3$.

If instead $T^4 v = 0$, then the equation $a_1 T^3 v + a_2 T^4 v = 0$ is simply $a_1 T^3 v = 0$, so $a_1 = 0$ since $T^3 v \neq 0$. The equation before this is then $a_2 T^3 v = 0$, so $a_2 = 0$; the equation before this is $a_3 T^3 v = 0$, so $a_3 = 0$; and the original equation becomes $a_4 T^3 v = 0$, so $a_4 = 0$. (Recall that in this case $T^4 v = 0$, so all of the $T^4 v$ terms in each equation are missing.) Thus, the only coefficients satisfying

$$a_1 v + a_2 Tv + a_3 T^2 v + a_4 T^3 v = 0$$

are $a_1 = a_2 = a_3 = a_4 = 0$, (again, there is no $T^4 v = 0$ term), so v, Tv, T^2v, T^3v are linearly independent. This is still impossible since $\dim V = 3$, so we conclude that T^3 must have been 0 all along. $\square$

7. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transformation defined by

$$T(A) = A \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A.$$

Take it for granted that this is linear. Find the rank of T and a basis for the image of T.

Solution 1. Set $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then

$$T(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2c & 2a-2d \\ 0 & 2c \end{bmatrix} = c \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} + (a-d) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This shows that anything in the image of T is a linear combination of $\left[\begin{smallmatrix} -2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, so these two matrices span $\operatorname{im} T$. Since they are linearly independent (neither is a multiple of the other), they thus form a basis for the image, and hence $\operatorname{rank} T = 2$. $\square$

Solution 2. Here is another approach. A problem from Homework 8 showed that $\dim \ker T = 2$, so by the Rank-Nullity Theorem we have (since $\dim M_2(\mathbb{R}) = 4$):

$$4 = \operatorname{rank} T + 2,$$

so $\operatorname{rank} T = 2$. Thus any two linearly independent elements of the image of T will form a basis. Since

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix},$$

the matrices $\left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -2 & 2 \\ 0 & 2 \end{smallmatrix}\right]$ are in $\operatorname{im} T$, and since they are linearly independent, they form a basis as desired. $\square$