Math 291-1: Final Exam
Northwestern University, Fall 2020

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false, and provide justification for your answer.

(a) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^3$ are linearly independent over $\mathbb{C}$, then they are linearly independent over $\mathbb{R}$.

(b) If A, B, C are 2×2 matrices such that $ABC = 0$, then none of A, B, C are invertible.

(c) If $T : P_2(\mathbb{R}) \to \mathbb{R}^+$ is linear, where $\mathbb{R}^+$ is the vector space of positive real numbers from Problem 7 of Homework 6, then $T(0) = 1$.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Suppose A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Show that $A\mathbf{x} = \mathbf{b}$ has no solution if and only if $\operatorname{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \operatorname{rank} A + 1$, where $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is the $m \times (n+1)$ matrix whose first n columns are those of A and whose final column is $\mathbf{b}$.

3. (10 points) Suppose P is a 2×2 matrix of rank 1 such that $P^2 = P$ with the property that anything in $\operatorname{im} P$ is perpendicular to anything in $\ker P$. Show that P is the matrix of an orthogonal projection. (Hint: First determine the line onto which P should orthogonally project an arbitrary vector, and for this think about the effect which P has on something in $\operatorname{im} P$.)

4. (10 points) Suppose A and B are $n \times n$ matrices, and that A is row-equivalent to I. Show that AB is row-equivalent to B. (Careful: it is NOT true in general that $\mathrm{rref}(CD) = \mathrm{rref}(C)\,\mathrm{rref}(D)$.)

5. (10 points) Suppose U is a subspace of $P_n(\mathbb{R})$ with the property that whenever $p(x)$ is in U, we have that its derivative $p'(x)$ is also in U. If $x^n \in U$, show that $U = P_n(\mathbb{R})$.

6. (10 points) Suppose V is a 3-dimensional vector space and that $T : V \to V$ is a linear transformation such that $T^5 = 0$. Show that $T^3 = 0$. (You cannot just quote a remark you might have seen somewhere which says that this is true—you must prove it. You can, however, look around at old exam problems to get an idea for what to do. Note the exponent in T^5 is 5, not 4.)

7. (10 points) Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transformation defined by

$$T(A) = A \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A.$$

Take it for granted that this is linear. Find the rank of T and a basis for the image of T.