Math 291-1: Midterm 1 Solutions
Northwestern University, Fall 2020

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If A is an $m \times n$ matrix for which $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $m \leq n$.

(b) If A, B are 2×2 matrices such that $\text{rref}(A) = \text{rref}(B)$, then $A\mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $B\mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ have the same solutions.

Solution. (a) This is false. For instance, take A to be the 3×2 zero matrix, meaning that all of its entries are zero. Then *every* $\mathbf{x} \in \mathbb{R}^2$ satisfies $A\mathbf{x} = \mathbf{0}$, so that this equation does have infinitely many solutions. But A has more rows than columns, so $m > n$ in this case. (What *is* true when $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions is that the columns of A would have to be linearly dependent, but it is certainly possible to have n linearly dependent columns coming from $\mathbb{R}^m$ no matter than m and n are. As a modification of this question, if instead we assumed that $A\mathbf{x} = \mathbf{0}$ had exactly one solution $\mathbf{x} = \mathbf{0}$, then in fact $m \geq n$ would be true since $\text{rref}(A)$ would need to have a pivot in each column in that case, which requires at least as many rows as columns.)

(b) This is false. For instance, take $A = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$. Then $\text{rref}(A) = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right] = \text{rref}(B)$ (B is already in reduced form), but $\mathbf{x} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ satisfies $A\mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ while it does not satisfy $B\mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, so these two equations do not have the same solutions. $\square$

2. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^n$ are linearly independent. (This requires that $n \geq 4$, but this is not important for this problem.) Show that

$$\mathbf{v}_1,\ \mathbf{v}_1 - \mathbf{v}_2,\ \mathbf{v}_2 - \mathbf{v}_3,\ \mathbf{v}_3 - \mathbf{v}_4$$

are also linearly independent.

Proof. Suppose $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are scalars satisfying

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 - \mathbf{v}_2) + c_3(\mathbf{v}_2 - \mathbf{v}_3) + c_4(\mathbf{v}_3 - \mathbf{v}_4) = \mathbf{0}.$$

After rearranging, this can be written as

$$(c_1 + c_2)\mathbf{v}_1 + (c_3 - c_2)\mathbf{v}_2 + (c_4 - c_3)\mathbf{v}_3 - c_4\mathbf{v}_4 = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, the only linear combination of these which can result in $\mathbf{0}$ is the one where all coefficients are taken to be 0, so we must have

$$c_1 + c_2 = 0,\ c_3 - c_2 = 0,\ c_4 - c_3 = 0, \text{ and } c_4 = 0.$$

Since $c_4 = 0$, $c_4 - c_3 = 0$ implies that $c_3 = 0$, and then $c_3 - c_2 = 0$ implies that $c_2 = 0$, and then finally $c_1 + c_2 = 0$ implies $c_1 = 0$. Thus, the only scalars satisfying

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 - \mathbf{v}_2) + c_3(\mathbf{v}_2 - \mathbf{v}_3) + c_4(\mathbf{v}_3 - \mathbf{v}_4) = \mathbf{0}$$

are $c_1 = c_2 = c_3 = c_4 = 0$, which shows that $\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4$ are linearly independent. $\square$

3. Suppose A is a 2×2 *complex* matrix, meaning that it has complex numbers as entries. Show, using induction, that for any $n \geq 2$ and n complex vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^2$, we have

$$A(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = A\mathbf{v}_1 + \cdots + A\mathbf{v}_n.$$

The only thing you can take for granted is that the distributive property $a(b+c) = ab + ac$ for *real* numbers $a, b, c \in \mathbb{R}$ is true, so as a first step you should verify that this is also true for complex numbers. You **cannot** assume that multiplication of complex matrices by complex vectors is distributive since that it is exactly what you are asked to prove.

Proof. First, for complex numbers $z = a + ib, w = c + id, u = p + iq$ with $a, b, c, d, p, q \in \mathbb{R}$, we have:

$$\begin{aligned} z(w+u) &= (a+ib)([c+p] + i[d+q]) \\ &= a[c+p] - b[d+q] + i(a[d+q] + b[c+p]) \\ &= (ac + ap - bd - dq) + i(ad + aq + bc + bp) \end{aligned}$$

and

$$\begin{aligned} zw + zu &= (a+ib)(c+id) + (a+ib)(p+iq) \\ &= ac - db + i(ad + bc) + ap - bq + i(aq + bp) \\ &= (ac - cb + ap - bq) + i(ad + bc + aq + bp). \end{aligned}$$

Thus $z(w+u) = zw + zu$, so the distributive property holds for multiplication of complex numbers.

Now, write A and two complex vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^2$ as

$$A = \begin{bmatrix} z & w \\ u & v \end{bmatrix}, \ \mathbf{v}_1 = \begin{bmatrix} h_1 \\ k_1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} h_2 \\ k_2 \end{bmatrix}$$

where $z, w, u, v, h_1, k_1, h_2, k_2 \in \mathbb{C}$. Then using the distributive property for complex numbers, we get for the base case of $n = 2$ complex vectors:

$$\begin{aligned} A(\mathbf{v}_1 + \mathbf{v}_2) &= \begin{bmatrix} z & w \\ u & v \end{bmatrix} \begin{bmatrix} h_1 + h_2 \\ k_1 + k_2 \end{bmatrix} \\ &= \begin{bmatrix} z(h_1 + h_2) + w(k_1 + k_2) \\ u(h_1 + h_2) + v(k_1 + k_2) \end{bmatrix} \\ &= \begin{bmatrix} zh_1 + wk_1 + zh_2 + wk_2 \\ uh_1 + vk_1 + uh_2 + vk_2 \end{bmatrix} \\ &= \begin{bmatrix} zh_1 + wk_1 \\ uh_1 + vk_1 \end{bmatrix} + \begin{bmatrix} zh_2 + wk_2 \\ uh_2 + vk_2 \end{bmatrix} \\ &= \begin{bmatrix} z & w \\ u & v \end{bmatrix} \begin{bmatrix} h_1 \\ k_1 \end{bmatrix} + \begin{bmatrix} z & w \\ u & v \end{bmatrix} \begin{bmatrix} h_2 \\ k_2 \end{bmatrix} \\ &= A\mathbf{v}_1 + A\mathbf{v}_2. \end{aligned}$$

Suppose that for some $n \geq 2$ our claim holds for any n complex vectors, and take any $n+1$ complex vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in \mathbb{C}^2$. Then:

$$\begin{aligned} A(\mathbf{v}_1 + \cdots + \mathbf{v}_{n+1}) &= A[(\mathbf{v}_1 + \cdots + \mathbf{v}_n) + \mathbf{v}_{n+1}] \\ &= A(\mathbf{v}_1 + \cdots + \mathbf{v}_n) + A\mathbf{v}_{n+1} \\ &= A\mathbf{v}_1 + \cdots + A\mathbf{v}_n + A\mathbf{v}_{n+1} \end{aligned}$$

where the second line follows from the base case and the third from the induction hypothesis. Thus we conclude by induction that our claim holds for all $n \geq 2$. ∎

4. Consider the system of linear equations which corresponds to the following augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 3 \\ -2 & -3 & -4 & 0 & -6 & -5 \\ 0 & 1 & 2 & 1 & 6 & 2 \end{bmatrix}$$

Find, with justification, three nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^5$ such that all solutions $\mathbf{x}$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Note: the vectors you find will necessarily be linearly **dependent**, so if the vectors you come up with are independent, then something went wrong.

Proof. First, note that

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ -2 & -3 & -4 & 0 & -6 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$$

is true, so the given explicit vector is a solution of this system. Thus all solutions of this system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + \mathbf{x}_h$$

where $\mathbf{x}_h$ is a solution of the corresponding homogeneous equation. Row-reducing the augmented matrix corresponding to this homogeneous system gives the following:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 0 \\ -2 & -3 & -4 & 0 & -6 & 0 \\ 0 & 1 & 2 & 1 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 2 & 1 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

Thus we find that all solutions of the homogeneous system are of the following form:

$$\mathbf{x}_h = \begin{bmatrix} x_3 + 3x_5 \\ -2x_3 - 4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Set $\mathbf{v}_1, \mathbf{v}_2$ to be these two vectors on the right. So far we thus have that any solution of the given system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Now, set $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ (or any linear combination of $\mathbf{v}_1, \mathbf{v}_2$). Then we can write the solution $\mathbf{x}$ above as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3.$$

(The point is that since $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_1, \mathbf{v}_2$ already span the set of all solutions to the homogeneous system, including $\mathbf{v}_3$ in this list does not alter this span: $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. In the expression above, $\mathbf{v}_3$ is thus unnecessary, so including it with a coefficient of $c_3 = 0$ does not affect the value of $\mathbf{x}$.)

Alternatively, if we insist on a nonzero value of c_3, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, we can also write the original expression for $\mathbf{x}$ in terms of $\mathbf{v}_1, \mathbf{v}_2$ as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + (c_1 - 1)\mathbf{v}_1 + (c_2 - 1)\mathbf{v}_2 + (\mathbf{v}_1 + \mathbf{v}_2) = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + (c_1 - 1)\mathbf{v}_1 + (c_2 - 1)\mathbf{v}_2 + \mathbf{v}_3.$$

There are in fact many ways of incorporating $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ in the term describing solutions of the homogeneous equation, and all we need is one to get the result desired. $\square$

5. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^5$ form the columns of a matrix A for which there exists $\mathbf{b} \in \mathbb{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Show that there is a vector $\mathbf{v}_5 \in \mathbb{R}^5$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^5$. Hint: interpret this all in terms of reduced row-echelon forms.

Proof. $\square$