Math 291-1: Midterm 1
Northwestern University, Fall 2020

Name: ____________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If A is an $m \times n$ matrix for which $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $m \leq n$.

(b) If A, B are 2×2 matrices such that $\operatorname{rref}(A) = \operatorname{rref}(B)$, then $A\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $B\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ have the same solutions.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^n$ are linearly independent. (This requires that $n \geq 4$, but this is not important for this problem.) Show that

$$\mathbf{v}_1,\ \mathbf{v}_1 - \mathbf{v}_2,\ \mathbf{v}_2 - \mathbf{v}_3,\ \mathbf{v}_3 - \mathbf{v}_4$$

are also linearly independent.

3. (10 points) Suppose A is a 2×2 *complex* matrix, meaning that it has complex numbers as entries. Show, using induction, that for any $n \geq 2$ and n complex vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^2$, we have

$$A(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = A\mathbf{v}_1 + \cdots + A\mathbf{v}_n.$$

The only thing you can take for granted is that the distributive property $a(b + c) = ab + ac$ for *real* numbers $a, b, c \in \mathbb{R}$ is true, so as a first step you should verify that this is also true for complex numbers. You **cannot** assume that multiplication of complex matrices by complex vectors is distributive since that it is exactly what you are asked to prove.

4. (10 points) Consider the system of linear equations which corresponds to the following augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 3 \\ -2 & -3 & -4 & 0 & -6 & -5 \\ 0 & 1 & 2 & 1 & 6 & 2 \end{bmatrix}$$

Find, with justification, three nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^5$ such that all solutions $\mathbf{x}$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \\ -1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Note: the vectors you find will necessarily be linearly **dependent**, so if the vectors you come up with are independent, then something went wrong.

5. (10 points) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^5$ form the columns of a matrix A for which there exists $\mathbf{b} \in \mathbb{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Show that there is a vector $\mathbf{v}_5 \in \mathbb{R}^5$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^5$. Hint: interpret this all in terms of reduced row-echelon forms.