Math 291-1: Midterm 2 Solutions
Northwestern University, Fall 2020

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A is a nonzero 2×2 matrix such that $A^2 = A$, then $A = I$.

(b) If U is a subspace of a finite-dimensional space V and $\dim U = \dim V$, then $U = V$.

Solution. (a) This is false. Take $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for instance. Then $A^2 = A$ but $A \neq I$. More generally, the matrix of any orthogonal projection (A above gives orthogonal projection onto the x-axis) onto a line will be a counterexample.

(b) This is true. Take a basis $u_1, \ldots, u_k$ for U. Since $\dim U = \dim V$, a basis for V will also contain k elements, so $u_1, \ldots, u_k$ must already be a basis for V as well. This means that $U = V$, since both U and V are spanned by the same vectors. $\square$

2. Set $\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{v}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ where $\mathbf{e}_i \in \mathbb{R}^3$ is the vector with a 1 as the i-th entry and 0 elsewhere. Suppose A is a 3×3 matrix. Show that

$$A\mathbf{v}_1 \in \operatorname{span}(\mathbf{v}_1),\ A\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2),\ \text{and } A\mathbf{v}_3 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

if and only if A is upper-triangular. (Recall that an upper-triangular matrix is one where the entry in the i-th row and j-th column is zero for $i > j$. In the 3×3 case, this means that the entries in the 2nd row 1st column, 3rd row 1st column, and 3rd row 2nd column are all zero.)

3. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ be vectors which span $\mathbb{R}^4$, and suppose A is an 4×4 matrix such that

$$A\mathbf{v}_1 = \mathbf{v}_2,\ A\mathbf{v}_2 = 2\mathbf{v}_4,\ A\mathbf{v}_3 = 3\mathbf{v}_3,\ \text{and } A\mathbf{v}_4 = 4\mathbf{v}_1.$$

(a) Show that A is invertible by expressing A as a product of invertible matrices.

(b) Show that A is invertible by showing that the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Proof. (a) Since

$$A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & A\mathbf{v}_4 \end{bmatrix},$$

the information in the setup gives

$$A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_2 & 2\mathbf{v}_4 & 3\mathbf{v}_3 & 4\mathbf{v}_1 \end{bmatrix}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^4$, $\mathbf{v}_2, 2\mathbf{v}_4, 3\mathbf{v}_3, 4\mathbf{v}_1$ also span $\mathbb{R}^4$ since any linear combination of the first set can be written as a linear combination of the second set:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \tfrac{1}{4}a_1(4\mathbf{v}_1) + a_2\mathbf{v}_2 + \tfrac{1}{3}a_3(3\mathbf{v}_3) + \tfrac{1}{2}a_4(2\mathbf{v}_4).$$

By the Amazingly Awesome theorem, both the matrix with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as columns and the one with $\mathbf{v}_2, 2\mathbf{v}_4, 3\mathbf{v}_3, 4\mathbf{v}_1$ as columns are invertible, so multiplying on the right by the inverse of the first gives:

$$A = \begin{bmatrix} \mathbf{v}_2 & 2\mathbf{v}_4 & 3\mathbf{v}_3 & 4\mathbf{v}_1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}^{-1},$$

which expresses A as the product of invertible matrices. Hence A is invertible.

(b) Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = 0$. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^4$, we can write

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 \text{ for some } a_1, a_2, a_3, a_4 \in \mathbb{R}.$$

Then

$$\mathbf{0} = A\mathbf{x} = a_1 A\mathbf{v}_1 + a_2 A\mathbf{v}_2 + a_3 A\mathbf{v}_3 + a_4 A\mathbf{v}_4 = a_1\mathbf{v}_2 + 2a_2\mathbf{v}_4 + 3a_3\mathbf{v}_3 + 4a_4\mathbf{v}_1.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^4$ and $\mathbb{R}^4$ is 4-dimesional, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are also linearly independent. Thus in the equation above we must have

$$a_1 = 0,\ 2a_2 = 0,\ 3a_3 = 0,\ \text{and } 4a_4 = 0,$$

which implies $a_1 = a_2 = a_3 = a_4 = 0$. Thus

$$\mathbf{x} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0},$$

so the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and hence A is invertible by the Amazingly Awesome Theorem. □

4. Suppose V is a vector space over $\mathbb{K}$ and that U is a subspace of V. Suppose further that $x, y \in V$ are elements such that $2x + 3y \in U$. If $4x + 9y \in U$, show that $x \in U$ and $y \in U$.

Proof. Since U closed under scalar multiplication and $2x + 3y \in U$, we have

$$-2(2x + 3y) = -4x - 6y \in U.$$

Since U is closed under addition and $4x + 9y \in U$, we also have

$$(4x + 9y) + (-4x - 6y) = 3y \in U.$$

Then $\frac{1}{3}(3y) = y$ is also in U. Moreover, $-3y \in U$ so

$$(2x + 3y) + (-3y) = 2x \in U,$$

which implies $\frac{1}{2}(2x) = x$ is in U as well. □

5. Let $B \in M_3(\mathbb{R})$ and let U be the set of all 3×3 matrices which commute with B:

$$U = \{A \in M_3(\mathbb{R}) \mid AB = BA\}.$$

(a) Show that U is a subspace of $M_3(\mathbb{R})$.

(b) Find a basis for U in the case where $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.