Math 291-1: Midterm 2
Northwestern University, Fall 2020

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A is a nonzero 2×2 matrix such that $A^2 = A$, then $A = I$.

(b) If U is a subspace of a finite-dimensional space V and $\dim U = \dim V$, then $U = V$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Set $\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{v}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ where $\mathbf{e}_i \in \mathbb{R}^3$ is the vector with a 1 as the i-th entry and 0 elsewhere. Suppose A is a 3×3 matrix. Show that

$$A\mathbf{v}_1 \in \mathrm{span}(\mathbf{v}_1),\ \ A\mathbf{v}_2 \in \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2),\ \text{ and } A\mathbf{v}_3 \in \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

if and only if A is upper-triangular. (Recall that an upper-triangular matrix is one where the entry in the i-th row and j-th column is zero for $i > j$. In the 3×3 case, this means that the entries in the 2nd row 1st column, 3rd row 1st column, and 3rd row 2nd column are all zero.)

3. (10 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ be vectors which span $\mathbb{R}^4$, and suppose A is an 4×4 matrix such that

$$A\mathbf{v}_1 = \mathbf{v}_2, \;\; A\mathbf{v}_2 = 2\mathbf{v}_4, \;\; A\mathbf{v}_3 = 3\mathbf{v}_3, \;\; \text{and } A\mathbf{v}_4 = 4\mathbf{v}_1.$$

(a) Show that A is invertible by expressing A as a product of invertible matrices.
(b) Show that A is invertible by showing that the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

4. (10 points) Suppose V is a vector space over $\mathbb{K}$ and that U is a subspace of V. Suppose further that $x, y \in V$ are elements such that $2x + 3y \in U$. If $4x + 9y \in U$, show that $x \in U$ and $y \in U$.

5. (10 points) Let $B \in M_3(\mathbb{R})$ and let U be the set of all 3×3 matrices which commute with B:

$$U = \{A \in M_3(\mathbb{R}) \mid AB = BA\}.$$

(a) Show that U is a subspace of $M_3(\mathbb{R})$.

(b) Find a basis for U in the case where $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.