Math 291-2: Final Exam
Northwestern University, Winter 2016

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A and B are diagonalizable 2×2 matrices, then $A + B$ is diagonalizable.

(b) There is no 3×3 matrix which transforms a sphere of radius 1 into a sphere of radius 3 and at the same time transforms a cube with edges of length 2 into one with edges of length 1.

(c) The limit $\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$ does not exist.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Suppose Q is an $n \times n$ matrix such that

$$Q\mathbf{x} \cdot Q\mathbf{y} = 4(\mathbf{x} \cdot \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Show that $\det Q = \pm 2^n$. Hint: Figure out how to relate Q to something which is orthogonal.

3. (10 points) For $n \geq 2$, let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = x^2 p''(x).$$

That is, T sends a polynomial $p(x)$ to $x^2 p''(x)$. Determine all eigenvalues and eigenvectors of T.

4. (10 points) Let A be an $n \times n$ symmetric matrix. Show that all eigenvalues of A are positive if and only if $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$.

5. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that for any $\mathbf{x} \in \mathbb{R}^n$, $\|f(\mathbf{x})\| \leq \|\mathbf{x}\|$ and

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\|\mathbf{h}\|} = \mathbf{0}.$$

Show that f is the zero function.

6. (10 points) Suppose $g : \mathbb{R}^n \to \mathbb{R}^m$ and $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$ are each differentiable and that

$$F(g(\mathbf{x}), \mathbf{x}) = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Write the Jacobian matrix of F at a point $\mathbf{y} \in \mathbb{R}^{m+n}$ as

$$DF(\mathbf{y}) = \begin{pmatrix} A(\mathbf{y}) & B(\mathbf{y}) \end{pmatrix}$$

where $A(\mathbf{y})$ is the $m \times m$ matrix consisting of the first m columns of $DF(\mathbf{y})$ and B is the $m \times n$ matrix consisting of the last n columns. If $\det A(g(\mathbf{x}), \mathbf{x}) \neq 0$ for each $\mathbf{x} \in \mathbb{R}^n$, show that

$$Dg(\mathbf{x}) = -A(g(\mathbf{x}), \mathbf{x})^{-1} B(g(\mathbf{x}), \mathbf{x}).$$

Hint: Express the function $\mathbf{x} \to F(g(\mathbf{x}), \mathbf{x})$ as a composition of differentiable functions.

7. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2. Fix a unit vector $\mathbf{u} \in \mathbb{R}^n$ and define the differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ by $g(\mathbf{x}) = D_{\mathbf{u}} f(\mathbf{x})$. Show that the maximal directional derivative of g at $\mathbf{x} \in \mathbb{R}^n$ occurs in the direction of $D^2 f(\mathbf{x})\mathbf{u}$, which is the product of the Hessian matrix $D^2 f(\mathbf{x})$ and the vector $\mathbf{u}$.