Math 291-2: Midterm 1 Solutions
Northwestern University, Winter 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $A, B \in M_n(\mathbb{R})$ are orthogonal, then $A + B$ is orthogonal.

(b) If $A \in M_3(\mathbb{R})$ satisfies $\text{Vol}(A(P)) = \text{Vol}(P)$ for some parallelopiped P in $\mathbb{R}^3$ of nonzero volume, then the only eigenvalues of A are ± 1. (Here, $A(P)$ denotes the image of P under the transformation determined by A.)

Solution. (a) This is false. For instance, I and $-I$ are orthogonal but $I + (-I) = 0$ is not.

(b) This is false. The condition $\text{Vol}(A(P)) = \text{Vol}(P)$ says that the expansion factor $|\det A|$ is 1. Taking for instance

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives $\det A = 1$, satisfying the given requirement, but the eigenvalues are $2, \frac{1}{2}, 1$. □

2. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$ with the property that

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_1 + \cdots + (\mathbf{x} \cdot \mathbf{v}_n)\mathbf{v}_n \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Show that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal.

Proof. Since the given equality is true for all $\mathbf{x}$, it is in particular true for each $\mathbf{x} = \mathbf{v}_i$. This gives that for each i:

$$\mathbf{v}_i = (\mathbf{v}_i \cdot \mathbf{v}_1)\mathbf{v}_1 + \cdots + (\mathbf{v}_i \cdot \mathbf{v}_i)\mathbf{v}_i + \cdots + (\mathbf{v}_i \cdot \mathbf{v}_n)\mathbf{v}_n.$$

Since we also have

$$\mathbf{v}_i = 0\mathbf{v}_1 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_n$$

and the coefficients needed to express a given vector as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are unique (since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$), we must have

$$\mathbf{v}_i \cdot \mathbf{v}_1 = 0, \ldots, \mathbf{v}_i \cdot \mathbf{v}_i = 1, \ldots, \mathbf{v}_i \cdot \mathbf{v}_n = 0,$$

or in other words

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ if } i \neq j \text{ and } \mathbf{v}_i \cdot \mathbf{v}_i = 1 \text{ for all } i.$$

The first condition says that $\mathbf{v}_1, \ldots \mathbf{v}_n$ are orthogonal to one another, and the second says they all have length 1, so $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal as claimed. □

3. Suppose that A is a 2×2 matrix such that $|\det A| = 1$ and which preserves angles, meaning that the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $A\mathbf{x}$ and $A\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Show that A is orthogonal.

You may use the following facts without proof. First, the angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is characterized by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|},$$

and second, the area of the parallelogram with edges $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$.

Proof. Take $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ to be nonzero vectors. Then the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as that between $A\mathbf{x}$ and $A\mathbf{y}$, so

$$\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} = \frac{A\mathbf{x} \cdot A\mathbf{y}}{\|A\mathbf{x}\| \, \|A\mathbf{y}\|}$$

since both sides give the value $\cos\theta$ for the same angle θ. Now, the parallelogram $A(P)$ with edges $A\mathbf{x}$ and $A\mathbf{y}$ is the image of the parallelogram P with edges $\mathbf{x}$ and $\mathbf{y}$ under the transformation determined by A, so

$$\text{area } A(P) = |\det A|(\text{area } P) = \text{area } P.$$

Assuming $\sin\theta \neq 0$, since

$$\text{area } A(P) = \|A\mathbf{x}\| \, \|A\mathbf{y}\| \sin\theta \text{ and area } P = \|\mathbf{x}\| \, \|\mathbf{y}\| \sin\theta$$

for the same angle θ, the equality of these areas gives

$$\|A\mathbf{x}\| \, \|A\mathbf{y}\| = \|\mathbf{x}\| \, \|\mathbf{y}\| .$$

Thus the first equation we had above becomes

$$\mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot A\mathbf{y},$$

showing that A preserves dot products, at least when $\mathbf{x}$ and $\mathbf{y}$ are not parallel, which was needed to ensure that $\sin\theta \neq 0$ above.

To show that $\mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot A\mathbf{y}$ even when $\mathbf{x}$ and $\mathbf{y}$ are parallel, we need a modified argument. (This part was actually much trickier than I envisioned when I first came up with the problem. Kudos if you were able to figure it out!) Pick any $\mathbf{v} \in \mathbb{R}^2$ which is not a multiple of $\mathbf{x}$. Then $\mathbf{x}$ and $\mathbf{x} + \mathbf{v}$ are not parallel, so what we did above shows that

$$\mathbf{x} \cdot (\mathbf{x} + \mathbf{v}) = A\mathbf{x} \cdot A(\mathbf{x} + \mathbf{v}),$$

which gives

$$\mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{v} = A\mathbf{x} \cdot A\mathbf{x} + A\mathbf{x} \cdot A\mathbf{v}.$$

Since $\mathbf{x}, \mathbf{v}$ are not parallel, what we did above shows that $\mathbf{x} \cdot \mathbf{v} = A\mathbf{x} \cdot A\mathbf{v}$, so we get

$$\mathbf{x} \cdot \mathbf{x} = A\mathbf{x} \cdot A\mathbf{x}.$$

Hence A preserves the dot product of a vector with itself. (Actually, this implies already that A preserves length, so A is orthogonal, but let's finish showing it preserves all dot products nonetheless.) Thus if $\mathbf{x}$ and $\mathbf{y}$ are parallel, say $\mathbf{y} = c\mathbf{x}$ for some $c \in \mathbb{R}$, we have:

$$A\mathbf{x} \cdot A\mathbf{y} = A\mathbf{x} \cdot A(c\mathbf{x}) = c(A\mathbf{x} \cdot A\mathbf{x}) = c(\mathbf{x} \cdot \mathbf{x}) = \mathbf{x} \cdot (c\mathbf{x}) = \mathbf{x} \cdot \mathbf{y},$$

so A preserves all dot products and hence is orthogonal as claimed. □

4. Suppose that an $n \times n$ matrix M is of the form

$$M = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

where A is a $k \times k$ matrix, C is an $(n-k) \times (n-k)$ matrix, and the 0's denote zero matrices. Show that $\det M = (\det A)(\det C)$. Suggestion: In the case where A is invertible, first consider the possibility where $A = I_k$ and then think about how you can reduce the general case to this one. The case where A is not invertible is simpler.

Proof. If A is not invertible, the its columns are linearly dependent and hence the first k columns of M are also linearly dependent. Hence M is also not invertible so

$$\det M = 0 = 0(\det C) = (\det A)(\det C)$$

as claimed. Thus we can now assume that A is invertible. The sequence of row operations transforming A to I will transform M as follows:

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \to \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix}.$$

If this involved t row swaps and row scalings by nonzero factors $s_1, \ldots, s_\ell$, we get

$$\det I_k = (-1)^n s_1 \cdots s_\ell (\det A) \text{ and } \det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix} = (-1)^n s_1 \cdots s_\ell \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

These together give

$$\det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \text{ or } \det M = (\det A) \det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix}.$$

Now, a cofactor expansion along the k-th column of the final matrix on the right gives

$$\det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix} = (-1)^{n+n} 1 \det \begin{pmatrix} I_{k-1} & 0 \\ 0 & C \end{pmatrix}$$

since the only nonzero entry in this k-th column is the final 1 in the I_k term, and crossing out the row and column this is in gives a smaller sized identity matrix in the upper left portion. Repeatedly doing the same cofactor expansion along the column corresponding to the final column of the I piece gives a similar expression until at the end we are left with $\det C$ alone, or to phrase this a bit more succinctly: we may assume by induction that

$$\det \begin{pmatrix} I_{k-1} & 0 \\ 0 & C \end{pmatrix} = \det C,$$

and hence

$$\det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix} = (-1)^{n+n} 1 \det \begin{pmatrix} I_{k-1} & 0 \\ 0 & C \end{pmatrix} = \det C.$$

Thus we get

$$\det M = (\det A) \det \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix} = (\det A)(\det C)$$

as claimed. □

5. Let $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the linear transformation defined by

$$S(A) = A^T.$$

Find the eigenvalues of S and determine a basis for each eigenspace. (Just give a basis for each eigenspace, you do not have to prove that what you claim is a basis is indeed a basis.)

Solution 1. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $A \neq 0$ is an eigenvector of S if

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. Comparing the first entries gives $a = \lambda a$, so either $\lambda = 1$ or $a = 0$. If $\lambda = 1$, comparing the upper-right and lower-left entries in the two matrices above gives $b = c$, so A is of the form

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

This makes sense, since in order for A to be an eigenvector with eigenvalue 1 we need $A^T = A$, which says that A is symmetric. A basis for the eigenspace corresponding to 1 is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, if $\lambda \neq 1$, the previous equations require that $a = 0$. Comparing the lower right entries in

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then shows that $d = 0$ as well. Comparing the remaining entries gives

$$c = \lambda b \text{ and } b = \lambda c, \text{ so } c = \lambda^2 c.$$

If $c = 0$, then $b = 0$, so our matrix is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which is not an eigenvector. Hence $c \neq 0$, in which case $\lambda^2 = 1$, so $\lambda = \pm 1$. Then eigenvalues 1 was dealt with before, so we see that -1 is the only other eigenvalue. When $\lambda = -1$, $c = -b$ so our matrix is of the form

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

which makes sense since the eigenvector equation $A^T = -A$ in this case says that A is skew-symmetric. Hence a basis for the eigenspace corresponding to -1 is given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To summarize, the eigenvalues of S are ± 1, with eigenspace corresponding to 1 being the space of symmetric matrices, and the eigenspace corresponding to -1 the space of skew-symmetric matrices. (This wasn't part of the problem, but note that S is diagonalizable since we have found four linearly independent eigenvectors and $\dim M_2(\mathbb{R}) = 4$.) □

Solution 2. With respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_2(\mathbb{R})$, the matrix of S is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

since S sends the first and fourth basis elements to themselves and exchanges the second and third. The eigenvalues of S are the same as the eigenvalues of this matrix. The characteristic polynomial (if you work it out using a cofactor expansion) is:

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = (\lambda-1)^3(\lambda+1),$$

so the eigenvalues are ± 1.

For $\lambda = 1$, we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which has kernel spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 11 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The matrices which have these coordinator vectors relative to the given basis are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which agrees with the basis we found for E_1 previously.

For $\lambda = -1$, we get

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

which has kernel spanned by

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The matrix with this coordinator vector is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which agrees with the basis we found for E_{-1} in the first solution. □