Math 291-2: Midterm 1
Northwestern University, Winter 2016

Name: ____

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $A, B \in M_n(\mathbb{R})$ are orthogonal, then $A + B$ is orthogonal.

(b) If $A \in M_3(\mathbb{R})$ satisfies $\text{Vol}(A(P)) = \text{Vol}(P)$ for some parallelopiped P in $\mathbb{R}^3$ of nonzero volume, then the only eigenvalues of A are ± 1. (Here, $A(P)$ denotes the image of P under the transformation determined by A.)

2. (10 points) Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$ with the property that

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_1 + \cdots + (\mathbf{x} \cdot \mathbf{v}_n)\mathbf{v}_n \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Show that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal.

3. (10 points) Suppose that A is a 2×2 matrix such that $|\det A| = 1$ and which preserves angles, meaning that the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $A\mathbf{x}$ and $A\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Show that A is orthogonal.

You may use the following facts without proof. First, the angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is characterized by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|},$$

and second, the area of the parallelogram with edges $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$.

4. (10 points) Suppose that an $n \times n$ matrix M is of the form

$$M = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

where A is a $k \times k$ matrix, C is an $(n-k) \times (n-k)$ matrix, and the 0's denote zero matrices. Show that $\det M = (\det A)(\det C)$. Suggestion: In the case where A is invertible, first consider the possibility where $A = I_k$ and then think about how you can reduce the general case to this one. The case where A is not invertible is simpler.

5. (10 points) Let $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the linear transformation defined by

$$S(A) = A^T.$$

Find the eigenvalues of S and determine a basis for each eigenspace. (Just give a basis for each eigenspace, you do not have to prove that what you claim is a basis is indeed a basis.)