Math 291-3: Final Exam
Northwestern University, Spring 2016

Name: SOLUTIONS

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous everywhere except on the ellipse $2x^2 + 3y^2 = 4$, then

$$\int_{-5}^{5} \int_{-6}^{6} f(x,y)\, dx\, dy = \int_{-6}^{6} \int_{-5}^{5} f(x,y)\, dy\, dx.$$

(b) If $\mathbf{F}$ is C^1 on an open set $U \subseteq \mathbb{R}^2$ and $\operatorname{curl} \mathbf{F} = \mathbf{0}$ on U, then $\mathbf{F}$ is conservative on U.

(c) If $\mathbf{F}$ is a C^1 vector field on a smooth C^1 closed surface S, then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

(a) TRUE along any vertical or horizontal segment of $[-6,6] \times [-5,5]$ f is discontinuous at only a finite number of points, so Fubini's Thm applies.

(b) FALSE $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$ on $U = \mathbb{R}^2$ with $(0,0)$ removed is a counterexample

(c) TRUE Split S into two halves $S = S_1 + S_2$. Then $\partial S_1 = \partial S_2$ but with opposite orientation, so

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} + \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$= \int_{\partial S_1} \vec{F} \cdot d\vec{s} + \int_{-\partial S_1} \vec{F} \cdot d\vec{s} = 0.$$

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Consider

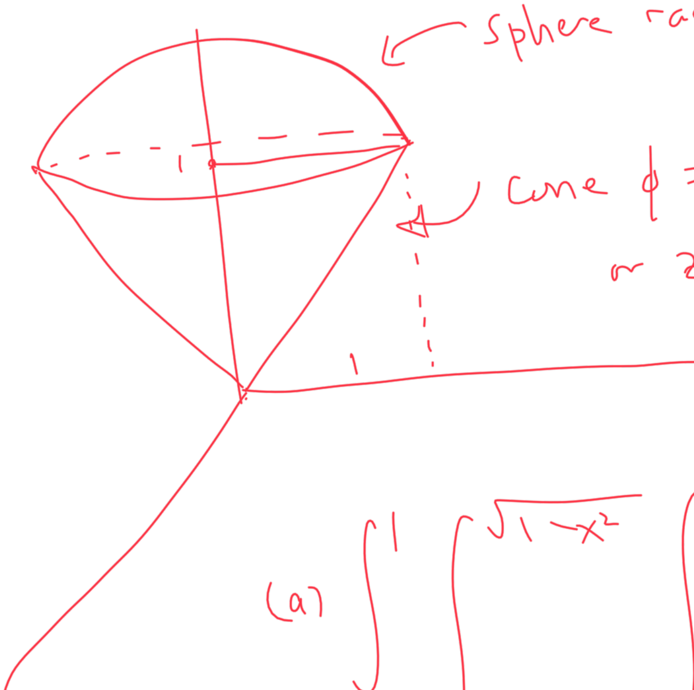
$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta.$$

(a) Rewrite this as a **single** iterated integral in rectangular coordinates.

(b) Rewrite this as a **sum** of iterated integrals in cylindrical coordinates.

The point is that you have to determine for yourself which orders of integration give a single integral in (a) and a sum of integrals in (b).

Sphere radius $\sqrt{2}$, $x^2+y^2+z^2=2$



cone $\phi = \pi/4$

or $z=\sqrt{x^2+y^2}$

(a) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x \, dz \, dy \, dx$$

(b) $$\int_0^{2\pi}\int_0^{1}\int_0^{z} r^2\cos\theta \, dr \, dz \, d\theta$$

$$+\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{2-z^2}} r^2\cos\theta \, dr \, dz \, d\theta$$

3. (10 points) Let D_R^n denote the disk of radius R centered at the origin in $\mathbb{R}^n$:

$$D_R^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq R^2\}.$$

(a) Show that

$$\mathrm{Vol}(D_1^n) = \iint_{D_1^2} \mathrm{Vol}\left(D^{n-2}_{\sqrt{1-x_1^2-x_2^2}}\right) dx_1\, dx_2$$

(b) Show that

$$\mathrm{Vol}(D_1^n) = \frac{2\pi}{n} \mathrm{Vol}(D_1^{n-2}) \text{ for } n \geq 3.$$

Check solution to

HW 3, Problem 10

4. (10 points) Prove the Fundamental Theorem of Line Integrals: if C is a smooth C^1 curve in $\mathbb{R}^n$ which starts at $\mathbf{p} \in \mathbb{R}^n$ and ends at $\mathbf{q} \in \mathbb{R}^n$, and f is a C^1 function on C, then

$$\int_C \nabla f \cdot d\mathbf{s} = f(\mathbf{q}) - f(\mathbf{p}).$$

Lecture Notes, Lecture 22

5. (10 points) Suppose $\mathbf{X} : D \to \mathbb{R}^3$ and $\mathbf{Y} : E \to \mathbb{R}^3$ are both parametrizations of a smooth C^1 surface in $\mathbb{R}^3$ which induce the same orientation. Suppose further that $\mathbf{Y} = \mathbf{X} \circ T$ for some C^1 bijective function $T : E \to D$ which has invertible Jacobian throughout E. Show that

$$\iint_E \mathbf{Y}(s,t) \cdot (\mathbf{Y}_s(s,t) \times \mathbf{Y}_t(s,t))\, d(s,t) = \iint_{D = T(E)} \mathbf{X}(u,v) \cdot (\mathbf{X}_u(u,v) \times \mathbf{X}_v(u,v))\, d(u,v).$$

You can take it for granted that the normal vector determined by $\mathbf{Y}$ at (s,t) is the one determined by $\mathbf{X}$ at $(u,v) = T(s,t)$ scaled by a factor of $\det DT(s,t)$.

Since $\vec{X}, \vec{Y}$ induce same orientation and

$$\vec{Y}_s \times \vec{Y}_t(s,t) = (\det DT(s,t))\left(\vec{X}_u \times \vec{X}_v(T(s,t))\right),$$

$\det DT(s,t)$ must be positive, so $|\det DT| = \det DT$.

Change of variables gives:

$$\text{right side} = \iint_E \vec{X}(T(s,t)) \cdot \left(\vec{X}_u \times \vec{X}_v(T(s,t))\right) |\det DT(s,t)|\, d(s,t)$$

$$= \iint_E \vec{X}(T(s,t)) \cdot \left[(\det DT(s,t))\left(\vec{X}_u \times \vec{X}_v(T(s,t))\right)\right] d(s,t)$$

$$= \iint_E \vec{Y}(s,t) \cdot \left(\vec{Y}_s \times \vec{Y}_t(s,t)\right) d(s,t)$$

$$= \text{left side}$$

6. (10 points) Suppose C is a simple, closed C^1 curve in the plane $2x + 3y - z = 3$, oriented counterclockwise when viewed from above. Show that the value of

$$\int_C (3e^{\cos x} + y - z)\, dx + (3x + e^{y^2} - 2z)\, dy + (x + y + z)\, dz$$

depends only on the area enclosed by C in the given plane.

Let D be the region on $2x+3y-z=3$ enclosed by C with upward orientation. By Stokes' Theorem, the given line integral equals

$$\iint_D \underbrace{\langle 3, -2, 2\rangle}_{\text{curl of } \langle \text{coefficient of } dx,\ \text{of } dy,\ \text{of } dz \rangle} \cdot d\vec{S}$$

unit normal vector is $\dfrac{\langle 2,3,-1\rangle}{\|\langle 2,3,-1\rangle\|}$, so

$$\iint_D \langle 3,-2,2\rangle \cdot d\vec{S} = \iint_D \frac{\langle 3,-2,2\rangle \cdot \langle 2,3,-1\rangle}{\|\langle 2,3,-1\rangle\|}\, dS$$

$$= \frac{\langle 3,-2,2\rangle \cdot \langle 2,3,-1\rangle}{\|\langle 2,3,-1\rangle\|} \cdot \text{Area}(D)$$

as claimed.

7. (10 points) Suppose E is a compact solid in $\mathbb{R}^3$ and that u is a C^2 function on E. If

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

on all of E and $u(x,y,z)=0$ for all $(x,y,z)\in\partial E$, show that $u(x,y,z)=0$ for all $(x,y,z)\in E$. Hint: Apply Gauss's Theorem to a well-chosen vector field.

Consider $u\nabla u = \langle uu_x, uu_y, uu_z\rangle$. Then

$$\iint_{\partial E} u\nabla u \cdot d\vec{S} = 0$$ since $u=0$ on ∂E.

By Gauss's Theorem, this surface integral equals

$$\iiint_E \operatorname{div}(u\nabla u)\, dV.$$

$$\operatorname{div}(u\nabla u) = (u_x)^2 + uu_{xx} + (u_y)^2 + uu_{yy} + (u_z)^2 + uu_{zz}$$

$$= \|\nabla u\|^2 + u\underbrace{(u_{xx}+u_{yy}+u_{zz})}_{=0}.$$

So $$0 = \iiint_E \|\nabla u\|^2\, dV.$$ Since $\|\nabla u\|^2$ is continuous and never negative, this means $\|\nabla u\|^2 = 0$.

So $\nabla u = \vec{0}$ and hence u is constant.

Thus $u=0$ on $\partial E \Rightarrow u=0$ everywhere.