Math 291-3: Final Exam
Northwestern University, Spring 2016

Name: ____

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous everywhere except on the ellipse $2x^2 + 3y^2 = 4$, then

$$\int_{-5}^{5} \int_{-6}^{6} f(x, y)\, dx\, dy = \int_{-6}^{6} \int_{-5}^{5} f(x, y)\, dy\, dx.$$

(b) If $\mathbf{F}$ is C^1 on an open set $U \subseteq \mathbb{R}^2$ and $\operatorname{curl} \mathbf{F} = \mathbf{0}$ on U, then $\mathbf{F}$ is conservative on U.

(c) If $\mathbf{F}$ is a C^1 vector field on a smooth C^1 closed surface S, then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

Problem	Score
1	
2	
3	
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7	
Total	

2. (10 points) Consider

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta.$$

(a) Rewrite this as a **single** iterated integral in rectangular coordinates.

(b) Rewrite this as a **sum** of iterated integrals in cylindrical coordinates.

The point is that you have to determine for yourself which orders of integration give a single integral in (a) and a sum of integrals in (b).

3. (10 points) Let D^n_R denote the disk of radius R centered at the origin in $\mathbb{R}^n$:

$$D^n_R = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \le R^2\}.$$

(a) Show that

$$\mathrm{Vol}(D^n_1) = \iint_{D^2_1} \mathrm{Vol}\left(D^{n-2}_{\sqrt{1-x_1^2-x_2^2}}\right)\, dx_1\, dx_2$$

(b) Show that

$$\mathrm{Vol}(D^n_1) = \frac{2\pi}{n}\, \mathrm{Vol}(D^{n-2}_1) \text{ for } n \ge 3.$$

4. (10 points) Prove the Fundamental Theorem of Line Integrals: if C is a smooth C^1 curve in $\mathbb{R}^n$ which starts at $\mathbf{p} \in \mathbb{R}^n$ and ends at $\mathbf{q} \in \mathbb{R}^n$, and f is a C^1 function on C, then

$$\int_C \nabla f \cdot d\mathbf{s} = f(\mathbf{q}) - f(\mathbf{p}).$$

5. (10 points) Suppose $\mathbf{X} : D \to \mathbb{R}^3$ and $\mathbf{Y} : E \to \mathbb{R}^3$ are both parametrizations of a smooth C^1 surface in $\mathbb{R}^3$ which induce the same orientation. Suppose further that $\mathbf{Y} = \mathbf{X} \circ T$ for some C^1 bijective function $T : E \to D$ which has invertible Jacobian throughout E. Show that

$$\iint_E \mathbf{Y}(s,t) \cdot (\mathbf{Y}_s(s,t) \times \mathbf{Y}_t(s,t))\, d(s,t) = \iint_D \mathbf{X}(u,v) \cdot (\mathbf{X}_u(u,v) \times \mathbf{X}_v(u,v))\, d(u,v).$$

You can take it for granted that the normal vector determined by $\mathbf{Y}$ at (s,t) is the one determined by $\mathbf{X}$ at $(u,v) = T(s,t)$ scaled by a factor of $\det DT(s,t)$.

6. (10 points) Suppose C is a simple, closed C^1 curve in the plane $2x + 3y - z = 3$, oriented counterclockwise when viewed from above. Show that the value of

$$\int_C (3e^{\cos x} + y - z)\,dx + (3x + e^{y^2} - 2z)\,dy + (x + y + z)\,dz$$

depends only on the area enclosed by C in the given plane.

7. (10 points) Suppose E is a compact solid in $\mathbb{R}^3$ and that u is a C^2 function on E. If

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

on all of E and $u(x, y, z) = 0$ for all $(x, y, z) \in \partial E$, show that $u(x, y, z) = 0$ for all $(x, y, z) \in E$. Hint: Apply Gauss's Theorem to a well-chosen vector field.