Math 300: Final Exam
Northwestern University, Winter 2019

Name: ____

1. (10 points) Give an example of each of the following with brief justification.
(a) A function $f : \mathbb{R} \to \mathbb{R}$ which is surjective but not injective.
(b) Nonempty subsets A_n of $\mathbb{Z}$ indexed by $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} A_n$ is empty.
(c) A subset of $\mathbb{R}$ not containing π but whose supremum is π.

Problem	Score
1	
2	
3	
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Total	

2. (10 points) Let A and B be the following sets:

$$A = \{n \in \mathbb{Z} \mid n = 7k - 17 \text{ for some } k \in \mathbb{Z}\}$$

and

$$B = \left\{n \in \mathbb{Z} \;\middle|\; n = 14k^3 + 4 \text{ for some } k \in \mathbb{Z}\right\}.$$

Show that $B \subseteq A$ and $A \neq B$.

3. (a) (5 points) Prove the following set containment:

$$[0,1) \subseteq \bigcup_{\epsilon>0} [0, 1-\epsilon)$$

(b) (5 points) Prove the following set containment:

$$\bigcap_{\epsilon>0} [0, 1+\epsilon] \subseteq [0,1]$$

4. Suppose $f : A \to B$ is a function and that $X, Y \subseteq A$.

(a) (5 points) Show that $f(X) - f(Y) \subseteq f(X - Y)$.

(b) (5 points) Show that if f is injective, then $f(X - Y) \subseteq f(X) - f(Y)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (2x - y - 1, x + 3).$$

(a) (5 points) Show that f is surjective.

(b) (5 points) Let S denote the subset of $\mathbb{R}^2$ consisting of all points on the line $y = x$. Show that the preimage $f^{-1}(S)$ of S under f is also a line by finding an explicit equation of this line. Just write the equation down after you work it out—you do not have to prove formally that $f^{-1}(S)$ equals the line you find by showing that each is a subset of the other.

6. (10 points) Define an equivalence relation on $\mathbb{R}^2$ by saying

$$(x, y) \sim (a, b) \text{ if } 2(y - b) = -3(x - a).$$

Show that the set of equivalence classes has the same cardinality as $\mathbb{R}$. (Careful: this is not asking about the cardinality of each equivalence class, but rather of the set whose **elements** are the equivalence classes.) Hint: Think about what each equivalence class looks like geometrically, and how you can characterize an entire class using a single real number.

7. (10 points) A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *periodic* if there exists $N \in \mathbb{N}$ such that

$$f(x + N) = f(x) \text{ for all } x \in \mathbb{N}.$$

(So, the values of f begin to repeat after some point: $f(N+1) = f(1), f(N+2) = f(2)$, etc.) Show that the set of periodic functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.

Hint: For a fixed $N \in \mathbb{N}$, consider the set X_N of functions from $\mathbb{N}$ to $\mathbb{N}$ satisfying the periodic condition $f(x+N) = f(x)$ for that specific N. Show that each X_N is countable, making use of the fact that such a function is characterized by the values of $f(1), f(2), \ldots, f(N)$ since other values beyond this are determined by the periodic condition.

8. (10 points) Show that the set of *all* functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable. Hint: A function $f : \mathbb{N} \to \mathbb{N}$ is determined by the infinite sequence containing its values:

$$(f(1), f(2), f(3), \ldots)$$

which is an element of $\mathbb{N}^\infty$. Thus, the set of functions from $\mathbb{N}$ to $\mathbb{N}$ has the same cardinality as $\mathbb{N}^\infty$, which can you take for granted. So, in other words, the problem is really to show that $\mathbb{N}^\infty$ is uncountable, which you must do directly without quoting any result from class.