# Math 306: Final Exam
## Northwestern University, Winter 2019

**Name**: ______________________________

**1.** (10 points) List the following things. No justification is needed, just list them.
(a) The self-conjugate (i.e. Young diagram is its own conjugate) integer partitions of 8.
(b) The derangements of the set [4]

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| Total | |

**2.** (10 points) Find the number of compositions of 16 into an odd number of even parts. Your answer should be a concrete number. Note: There will be too many such compositions to try to list them out by hand, so you need to find a better way of counting. You will need the fact that the number of compositions of $n$ into $k$ parts is $\binom{n-1}{k-1}$.

**3.** (10 points) Find explicit formulas for the Stirling numbers $S(n, 2)$ and $S(n, 3)$ of the second kind. Hint for $S(n, 3)$: Determine the value of $6S(n, 3) + 6S(n, 2) + 3S(n, 1)$ by thinking about what this counts in terms of functions on the set $[n]$, and use the formula you found for $S(n, 2)$.

**4.** (10 points) Justify the following recursive identity for the unsigned Stirling numbers:

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

**5.** (10 points) Justify the fact that

$$k!S(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

by interpreting both sides as counting the same thing. Hint: The first term in the sum on the right is $k^n$, which is the number of functions from $[n]$ to $[k]$.

**6.** (10 points) Suppose we arrange $n$ people in a line. We start by picking either the first person in the line and giving them a red hat or the first two people and given them both a blue hat or a green hat. (To be clear: if two people are picked, they both get the same color hat.) After we've done this, we then do the same thing for the shorter line consisting of the remaining people, and then again, and then again until we've gone through the entire line. At no point do we modify the order of people in the line. Find the number of ways in which this can all be done. Hint: The factorization $1 - x - 2x^2 = (1 + x)(1 - 2x)$ will be relevant.

**7.** (10 points) Suppose we have $n$ books, which we divide into two groups and place onto two shelves. We arrange the books on the first shelf in some order, and then we pick out an odd number of books from the second shelf. (The second shelf must have at least one book, but there is no restriction on the number of books the first shelf has.) Find the exponential generating function for the number of ways in which this can all be done.

**8.** (10 points) Let $a_n$ denote the number of permutations $\sigma$ of $[n]$ so that neither it nor its square $\sigma^2$ have any fixed points. Find the exponential generating function of the numbers $a_n$. Hint: What cycle lengths should be avoided in order to guarantee that neither $\sigma$ nor $\sigma^2$ have any fixed points?