Math 306: Midterm 1 Solutions
Northwestern University, Winter 2019

1. List the following things. No justification is needed, just list them.

(a) The compositions of 10 into an even number of even parts.

(b) The parenthetical expressions defining the 3-rd Catalan number, or, if you prefer, the paths in a 3×3 grid which define the 3-rd Catalan number.

Solution. (a) These compositions are:

$$8+2 \quad 6+4 \quad 4+2+2+2 \quad 2+4+2+2$$

$$2+2+4+2 \quad 2+2+2+4 \quad 4+6 \quad 2+8$$

(b) The parenthetical expressions are:

$$((())) \quad (())() \quad (()()) \quad ()(()) \quad ()()()$$

The paths, were U means "up" and R means "right" are:

$$RRRUUU \quad RRUURU \quad RRURUU \quad RURRUU \quad RURURU$$

$\square$

2. Do either (a) or (b). (You can do them both if you'd like for 2 points extra credit.)

(a) Let $n \geq 2$. Show that if we select $n+1$ integers from the set $[2n]$, there will be two among them so that one is a multiple of the other.

(b) Let $n \geq 3$. Show that any convex n-gon can be split up into $n-2$ triangles by drawing line segments which connect vertices. (A *convex* polygon is one where these line segments lie within the polygon.)

Solution. (a) Create the following groupings (i.e. "boxes/pigeonholes") of integers:

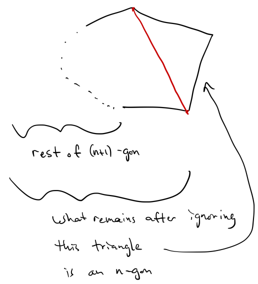
$$\{1, 2, 4, 8, 16, \ldots\}, \{3, 6, 12, 24, \ldots\}, \{5, 10, 20, \ldots, \}, \{7, 14, 24, \ldots\}, \ldots, \{2n-1\}$$

where the first consists of all powers of 2, the second consists of all numbers obtained by taking 3 times a power of 2, the third takes 5 times powers of 2, and so on. To be clear, we have one grouping for each odd integer, and the terms in that grouping are all things obtained by taking that odd integer times powers of 2. Since any integer is expressible as something odd times a power of 2, all elements of $\{1, 2, \ldots, 2n\}$ occur in exactly one of these groupings. There are n groupings since there are n odd numbers between 1 and $2n$, so with $n+1$ numbers chosen the Pigeonhole Principle guarantees that two chosen integers fall within the same grouping. This gives two integers of the form

$$m2^k \text{ and } m2^\ell$$

where $k, \ell \geq 0$ and m is the same odd integer in both. Then the larger of these, say $\ell > k$, is a multiple of the smaller since $m2^\ell = m2^k(2^{\ell-k})$.

(b) The base case $n = 3$ is a single triangle, so there is nothing to do since this already consists of $3 - 2 = 1$ triangle. Assuming we can do this for any n-gon (for some $n \geq 3$), we need to build up the case of an $(n+1)$-gon. Given some $(n+1)$-gon, we must thus find the "induction hypothesis" case of an n-gon hiding within our $(n+1)$-gon. But observe that if we connect two vertices like so:



so we connect two vertices which happen to be adjacent to the *same* vertex, we end up dividing our original $(n+1)$-gon into an n-gon and a triangle; the resulting n-gon can be broken up into $n-2$ triangles by the induction hypothesis, so in the end we get that our original $(n+1)$-gon is broken up into $(n-2)+1 = (n+1)-2$ triangles as required. Note that convexity was used to guarantee that the segment we introduced to connect the two vertices above does indeed result in an n-gon and a triangle. $\square$

3. Let $n \geq 4$. Determine the number of subsets of $[n]$ which contain at least one of 1 or 2, and *at the same time* exactly one of 3 or 4.

Solution. First, for a subset which contains 1 but not 2 and exactly one of 3 or 4, the remaining elements come from a subset of $\{5, 6, \dots, n\}$, and there are 2^{n-4} such subsets. Thus there are $2^{n-4} + 2^{n-4} = 2^{n-3}$ subsets containing 1, not 2, and exactly one of 3 or 3. Along the same lines, by exchanging the roles of 1 and 2 we get that there are 2^{n-3} subsets which contain 2, not 1, and exactly one of 3 or 4.

Finally, for a subset which contains both 1 and 2, and exactly one of 3 or 4, there are again $2^{n-4} + 2^{n-4} = 2^{n-3}$ possibilities since the other elements come from a subset of $\{5, 6, \dots, n\}$. In total we thus get

$$2^{n-3} + 2^{n-3} + 2^{n-3} = 2^{n-2} + 2^{n-3}$$

subsets of $[n]$ which contain 1 or 2 and exactly one of 3 or 4. $\square$

4. Justify the following identity by interpreting both sides as counting the same thing.

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2$$

Hint: Think about a construction involving picking a committee from a group of $2n$ people. Thinking about the identity $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$ first might help.

Solution. Take $2n$ people in a line:

$$1 \quad 2 \quad 3 \quad \cdots \quad n \quad n+1 \quad \cdots \quad 2n$$

Both sides of the given identity count the number of ways of forming a committee of n people from these $2n$ and then picking a president of the committee who is required to come from the first half $1, 2, 3, \dots, n$. We can first pick the president in n ways, and then the rest of the $n-1$ committee members from the remaining $2n-1$ people, which can be done in $\binom{2n-1}{n-1}$ ways. Thus we get

$$n\binom{2n-1}{n-1}$$

as one expression for the number of ways of forming the required committee.

Alternatively, if k people (where $k \geq 1$) in the committee come from the first half of people, and the other $n-k$ from the second half, we have $\binom{n}{k}\binom{n}{n-k}$ possible ways of forming the committee, and then k ways of picking the president since k people were chosen from the half the president must come from. This gives

$$k\binom{n}{k}\binom{n}{n-k}$$

possibilities when k people come from the first half. Using the fact that $\binom{n}{n-k} = \binom{n}{k}$ and summing over the possible values of k gives

$$\sum_{k=1}^{n} k\binom{n}{k}^2$$

ways of forming the required committee, so this expression must equal the one we found before. □

5. Let a_n denote the number of compositions of n into odd parts. (That is, parts each of which are odd, *not* necessarily an odd number of parts overall.) Compute a_1, a_2, a_3, a_4, a_5 and determine, with justification, a recursive identity for a_n in terms of some (which ones to use is up to you) a_k for smaller k.

Solution. The allowable compositions for $n = 1, 2, 3, 4, 5$ are respectively:

$$\begin{aligned}
&1\\
&1+1\\
&3,\ 1+1+1\\
&3+1,\ 1+3,\ 1+1+1+1\\
&5,\ 3+1+1,\ 1+3+1,\ 1+1+3,\ 1+1+1+1+1
\end{aligned}$$

Thus we get $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5$, which you might recognize as the first few terms in the Fibonacci sequence.

Indeed, we claim that the numbers a_n in general satisfy the recursion

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

There are two types of compositions of n into odd parts we can consider: those where the final part is 1, and those where the final part is at least 3. For those of the first type, removing the final part of 1 gives a composition of $n-1$ into odd parts, where we get odd parts since these were present in the original composition of n into odd parts. Conversely, given a composition of $n-1$ into odd parts, adding a new final part of 1 gives a valid composition of n, so

$$(\#\text{ valid compositions of } n \text{ with final part } 1) = (\#\text{ valid compositions of } n-1) = a_{n-1}$$

For a valid composition of n with final part at least 3, subtracting 2 from this final part gives a valid composition of $n-2$ (still with odd parts), and given a composition of $n-2$ into odd parts, adding 2 to the final part gives a composition of n into odd parts with final part at least 3, so:

$$(\#\text{ valid compositions of } n \text{ with final part at least } 3) = (\#\text{ valid compositions of } n-2) = a_{n-2}$$

Adding these values together gives all possible valid compositions of n, so we do get the recursive identity $a_n = a_{n-1} + a_{n-2}$ as claimed. □