Math 306: Midterm 2 Solutions
Northwestern University, Winter 2019

1. List out all the partitions (set partitions in the first case and integer partitions in the second) which are counted by the following numbers.
(a) $S(4,2)$
(b) $p(6)$

Solution. (a) The partitions of $\{1,2,3,4\}$ into two non-empty subsets are:

$$\{1\},\{2,3,4\} \quad \{2\},\{1,3,4\} \quad \{3\},\{1,2,4\} \quad \{4\},\{1,2,3\}$$

$$\{1,2\},\{3,4\} \quad \{1,3\},\{2,4\} \quad \{1,4\},\{2,3\}$$

(b) The partitions of 6 are:

$$6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1$$

$$2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1$$

□

2. Show that the number of integer partitions of n in which the two largest parts are equal is $p(n)-p(n-1)$, where $p(k)$ denotes the number of all partitions of k. You cannot take any fact derived in the book, or in class, or elsewhere for granted, and must justify any result you make use of. For instance, you cannot use without justification any fact about the number of partitions of n where all parts are of at least a certain size.

Solution. By taking conjugates of Young diagrams, the number of partitions of n in which the two largest parts are equal is the same as the number of partitions in which all parts are at least of size 2. To count partitions with all parts at least 2, we can take the number of all partitions and subtract way those which have a part equal to 1:

$$p(n) - (\#\text{ with a part equal to } 1)$$

But for a partition with a part equal to 1, removing this part gives a partition of $n-1$, and conversely given any partition of $n-1$ we can add a new part of 1 to obtain a partition of n with a part equal to 1. Thus, the number of partitions of n with a part equal to 1 is the same as the number of all partitions of $n-1$, which is $p(n-1)$. Hence the number of partitions of n in which the two largest parts are equal, which is the same as the number where all parts are of size at least 2, is $p(n)-p(n-1)$. □

3. Compute the unsigned Stirling number $c(9,3)$. You should express your answer as a concrete sum, but you do not have to simplify factorials, products, nor quotients, so that your answer does not have to be a single explicit number. Hint: What types of cycles do you need to describe the permutations counted by $c(9,3)$?

Solution. The possible cycle lengths for a permutation of $[9]$ which is to consist of three cycles overall are:

$$\text{(7-cycle)(1-cycle)(1-cycle)} \quad \text{(6-cycle)(2-cycle)(1-cycle)} \quad \text{(5-cycle)(3-cycle)(1-cycle)}$$

(5-cycle)(2-cycle)(2-cycle)   (4-cycle)(4-cycle)(1-cycle)   (4-cycle)(3-cycle)(2-cycle)

and   (3-cycle)(3-cycle)(3-cycle)

The number of permutations of each of these types respectively are:

$$\frac{9!}{7 \cdot 2!} \qquad \frac{9!}{6 \cdot 2} \qquad \frac{9!}{5 \cdot 3}$$

$$\frac{9!}{5 \cdot 2^2 \cdot 2!} \qquad \frac{9!}{4^2 \cdot 2!} \qquad \frac{9!}{4 \cdot 3 \cdot 2}$$

$$\text{and} \quad \frac{9!}{3^3 \cdot 3!}$$

Thus we get that

$$c(9,3) = \frac{9!}{7 \cdot 2!} + \frac{9!}{6 \cdot 2} + \frac{9!}{5 \cdot 3} + \frac{9!}{5 \cdot 2^2 \cdot 2!} + \frac{9!}{4^2 \cdot 2!} + \frac{9!}{4 \cdot 3 \cdot 2} + \frac{9!}{3^3 \cdot 3!}$$

□

4. Find a formula for the number of set partitions of $[3m]$ in which no multiple of 3 is in a singleton block, meaning the partition does not contain $\{3\}$, nor $\{6\}$, nor $\{9\}$, etc. Hint: First find a summation formula for the number of partitions which *do* contain such a singleton block. You will need to express your answer in terms of Bell numbers $B(k)$, which if you recall describe the number of all set partitions of $[k]$.

Solution. First we count the number of partitions in which a multiple of 3 *is* in a singleton block. If 3ℓ is to be in a singleton block, then the $3m-1$ remaining elements of $[3m]$ can be partitioned in any of $B(3m-1)$ many ways. There are m choices for the number 3ℓ, so we get at first glance

$$\binom{m}{1} B(3m-1)$$

partitions which have at least one multiple of 3 in a singleton block.

But this value over-counts those partitions which have at least *two* multiples of 3 in singleton blocks. If two multiples of 3 are to be in singleton blocks, the remaining $3m-2$ elements of $[3m]$ can be partiitoned in $B(3m-2)$ many ways, so with $\binom{m}{2}$ choices for the two multiples of 3 which are to be in singleton blocks we get

$$\binom{m}{2} B(3m-2)$$

partitions which have at least two multiples of 3 in singleton blocks, so we must subtract this number away from the number we found for partitions with at least one singleton block containing a multiple of 3.

But this now subtracts away too many times those partitions with at least three multiples of 3 in singleton blocks, so these have to added back in, and then we must subtract the number with at least four multiples of 3 in singleton blocks, and so on. In general, there are

$$\binom{m}{k} B(3m-k)$$

partitions with at least k multiples of 3 in singleton blocks, so by inclusion/exclusion the number of partitions with at least one multiple of 3 in a singleton block is:

$$\binom{m}{1}B(3m-1) - \binom{m}{2}B(3m-2) + \binom{m}{3}B(3m-3) - \cdots = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}B(3m-k).$$

Thus the number of partitions in which no multiple of 3 is in singleton block is:

$$B(3m) - \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}B(3m-k) = \sum_{k=0}^{m}(-1)^{k}\binom{m}{k}B(3m-k).$$

□

5. Say we have n people arranged in a line, which we split into at most two groups by placing a division point somewhere in the line; so, we never mix up the order of the people, and if one person ends up in the first group then all people before that person in the line also go into the first group. We allow for the possibility that one group is empty, which is why we get at most two groups. To each person in the first group we give either a red or blue hat, and to each person in the second group we give either a green, white, or black hat.

(a) Explain why the ordinary generating function for the number of ways of carrying out this process is

$$\frac{1}{(1-2x)(1-3x)}$$

(b) Find an explicit formula, which does not use a summation, for the number of ways of carrying out this process. (You should be able to do this part even if you can't figure out the first part.)

Solution. (a) With n people, there are 2^n ways of distributing red or blue hats, so the ordinary generating function for this the number of ways of distributing these two hats is

$$A(x) = \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(2x)^n = \frac{1}{1-2x}$$

With n people, there are 3^n ways of distributing green, white, or black hats, so the ordinary generating function for this the number of ways of distributing these three hats is

$$B(x) = \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}(3x)^n = \frac{1}{1-3x}$$

Thus, since we can structure our procedure as "form the two groups by putting a split among our people, so the thing counted by $A(x)$ on the first group and the thing counted by $B(x)$ on the second group", the ordinary generating function for this entire process is

$$A(x)B(x) = \frac{1}{1-2x}\cdot\frac{1}{1-3x} = \frac{1}{(1-2x)(1-3x)}$$

(b) We have:

$$\frac{1}{(1-2x)(1-3x)} = -\frac{2}{1-2x} + \frac{3}{1-3x},$$

so

$$A(x)B(x) = -2\sum_{n=0}^{\infty} 2^n x^n + 3\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}(3^{n+1}-2^{n+1})x^n.$$

Thus the number of ways of carrying out our process (the coefficient of x^n) is $3^{n+1}-2^{n+1}$. □