MATH 320-1: Final Exam Solutions
Northwestern University, Fall 2023

1. (15 points) Give an example of each of the following. You do not have to justify your answer.
(a) A Cauchy sequence in $(2, 3)$ which does not converge to an element of $(2, 3)$.
(b) A function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 2} f(x)^2$ exists but $\lim_{x\to 2} f(x)$ does not.
(c) A continuous function on $\mathbb{R}$ that is not differentiable at 2.
(d) A function $f : [0, 1] \to \mathbb{R}$ which is not integrable but for which $f(x)^2$ is integrable.
(e) An integrable function on $[2, 3]$ which is not continuous on $[2, 3]$.

2. Suppose the sequence (x_n) converges to x and that $-2 < x_n < 3$ for all $n \geq 1000$. Show that $-2 \leq x \leq 3$. You cannot simply quote the fact that convergence of sequences preserves non-strict inequalities since the goal is to prove exactly this in this special case.

Proof. If $x < -2$, then for $\epsilon_1 = -2 - x > 0$, all elements of the interval $(x - \epsilon_1, x + \epsilon_1)$ are strictly less than -2 since $x + \epsilon_1 < -2$. But since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon_1, \text{ or equivalently } x_n \in (x - \epsilon_1, x + \epsilon_1) \text{ for } \geq N_1.$$

For $n \geq \max\{N_1, 1000\}$, this contradicts the assumption $-2 < x_n$, so we must have $-2 \leq x$.

If $3 < x$, then for $\epsilon_2 = x - 3 > 0$, all elements of the interval $(x - \epsilon_2, x + \epsilon_2)$ are strictly larger than 3 since $3 < x - \epsilon_2$. But since $x_n \to x$, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon_2, \text{ or equivalently } x_n \in (x - \epsilon_2, x + \epsilon_2) \text{ for } \geq N_2.$$

For $n \geq \max\{N_2, 1000\}$, this contradicts the assumption $x_n < 3$, so we must have $x \leq 3$. Hence $-2 \leq x \leq 3$ as claimed. □

3. Suppose $f : (0, \infty) \to \mathbb{R}$ is uniformly continuous and that (x_n) and (y_n) are two sequences in $(0, \infty)$ that converge to 0. Show that the sequences $(f(x_n))$ and $(f(y_n))$ converge, and that the thing to which they converge is the same. You cannot use the fact that uniformly continuous functions can be extended to endpoints since this problem is essentially the proof of this fact. You can, however, use other properties of uniformly continuous functions.

Proof. First, since (x_n) and (y_n) converges, they are each Cauchy, so $(f(x_n))$ and $(f(y_n))$ are each Cauchy since uniformly continuous functions send Cauchy sequences to Cauchy sequences. Denote by L the number to which $(f(x_n))$ converges. We claim that $(f(y_n))$ also converges to L.

Indeed, let $\epsilon > 0$ and using uniform continuity pick $\delta > 0$ such that

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Pick $N \in \mathbb{N}$ such that

$$|x_n - 0| < \frac{\delta}{2}, \ |y_n - 0| < \frac{\delta}{2}, \text{ and } |f(x_n) - L| < \frac{\epsilon}{2},$$

which is possible using convergence of (x_n) and (y_n) to 0 and of $(f(x_n))$ to L. (There's some picking a maximum of three N's going on here.) Then if $n \geq N$, we have

$$|x_n - y_n| \leq |x_n - 0| + |y_n - 0| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and hence

$$|f(x_n) - f(y_n)| < \frac{\epsilon}{2}.$$

Thus for $n \geq N$, we get

$$|f(y_n) - L| < |f(y_n) - f(x_n)| + |f(x_n) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $(f(y_n))$ also converges to L as desired.

Alternatively, you can use the fact that if f is uniformly continuous and $x_n - y_n \to 0$, then $f(x_n) - f(y_n) \to 0$ to skip some of the work above. Since $x_n \to 0$ and $y_n \to 0$, we do have

$$x_n - y_n \to 0 - 0 = 0$$

and thus $f(x_n) - f(y_n) \to 0$ as well. Thus, if $f(x_n) \to L$, then

$$|f(y_n) - L| \leq |f(y_n) - f(x_n)| + |f(x_n) - L|$$

implies, by the squeeze theorem, that the left side converges to 0 as $n \to \infty$ since the two terms on the right do, so $f(y_n) \to L$ as claimed. $\square$

4. Fix $a \in \mathbb{R}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and differentiable at all $x \neq a$. If $\lim_{x\to a} f'(x)$ exists, show that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists and that f' is continuous at a. You cannot use L'Hopital's rule, which we did not cover in this course, to show this limit exists.

Proof. For any $x \neq a$, by the mean value theorem there exists c_x between x and a such that

$$f(x) - f(a) = f'(c_x)(x - a), \text{ or } \frac{f(x) - f(a)}{x - a} = f'(c_x).$$

Thus

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} f'(c_x).$$

Since c_x is between x and a, as $x \to a$ we also have $c_x \to a$, so the assumption that $\lim_{x\to a} f'(x)$ exists implies that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} f'(c_x)$$

exists. Hence $f'(a)$ exists, and

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} f'(c_x) = \lim_{x\to a} f(x),$$

which means that f' is continuous at a. $\square$

5. Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Show that

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\} \leq \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

You can take the relation between the upper and lower sums of f with respect to a partition and a refinement of that partition for granted. (Recall that P' is a refinement of P if P' is obtained from P by introducing more partition points.)

Proof. For any partitions P and Q of $[a,b]$, $P \cup Q$ is a refinement of both P and Q, and thus

$$L(f,P) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,Q).$$

Thus for fixed P, $L(f,P)$ is a lower bound on the set of all upper sums, and hence

$$L(f,P) \leq \inf\{U(f,Q) \mid Q \text{ is a partition of } [a,b]\}$$

for all P. But this in turns means that the right side is an upper bound on the set of all lower sums, so

$$\sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\} \leq \inf\{U(f,Q) \mid Q \text{ is a partition of } [a,b]\}$$

as claimed. $\square$

6. Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 5 + e^{3x} - \sin(\cos 4x) & \text{if } x \neq \frac{1}{n} \text{ for any } n \in \mathbb{N} \\ 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Show that g is integrable on $[0,1]$. You cannot use the Riemann-Lebesgue Theorem from the last day of class. Hint: For any positive c, there are only finitely many n such that $c < \frac{1}{n}$.

Proof. Note first that 35 is a bound on g over $[0,1]$ since

$$|5 + e^{3x} - \sin(\cos 4x)| \leq 5 + e^{3x} + |\sin(\cos(4x))| \leq 5 + e^3 + 1 < 5 + 3^3 + 1 \leq 35$$

for all $x \in [0,1]$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be the largest positive integer for which

$$\frac{\epsilon}{70} < \frac{1}{N}.$$

$\square$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 + 5e^{-1/x} & x > 0 \\ 0 & x \leq 0, \end{cases}$$

which is integrable on any closed interval, and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_0^{x^3} f(t)\sin(t)\,dt.$$

Show that F is continuously differentiable $\mathbb{R}$ but not twice differentiable on $\mathbb{R}$.