Northwestern University

MATH 320-1 Final Exam
Fall Quarter 2023
December 6, 2023

First name: ____________________ Last name: ____________________

There are seven problems. Be clear in your work about what is scratch work and what is what you actually want graded. There are two extra pages on the back; if you use these pages for work you want graded, be sure to indicate to which problem the work corresponds.

1. Give an example of each of the following. You do not have to justify your answer.
   (a) (3 points) A Cauchy sequence in $(2,3)$ which does not converge to an element of $(2,3)$.

   (b) (3 points) A function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 2} f(x)^2$ exists but $\lim_{x\to 2} f(x)$ does not.

   (c) (3 points) A continuous function on $\mathbb{R}$ that is not differentiable at 2.

   (d) (3 points) A function $f : [0,1] \to \mathbb{R}$ which is not integrable but for which $f(x)^2$ is integrable.

   (e) (3 points) An integrable function on $[2,3]$ which is not continuous on $[2,3]$.

2. (10 points) Suppose the sequence (x_n) converges to x and that $-2 < x_n < 3$ for all $n \geq 1000$. Show that $-2 \leq x \leq 3$. You cannot simply quote the fact that convergence of sequences preserves non-strict inequalities since the goal is to prove exactly this in this special case.

3. Suppose $f : (0, \infty) \to \mathbb{R}$ is uniformly continuous and that (x_n) and (y_n) are two sequences in $(0, \infty)$ that converge to 0. Show that the sequences $(f(x_n))$ and $(f(y_n))$ converge, and that the thing to which they converge is the same. You cannot use the fact that uniformly continuous functions can be extended to endpoints since this problem is essentially the proof of this fact. You can, however, use other properties of uniformly continuous functions.

4. (10 points) Fix $a \in \mathbb{R}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at all $x \neq a$. If $\lim_{x \to a} f'(x)$ exists, show that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and that f' is continuous at a. You cannot use L'Hopital's rule, which we did not cover in this course, to show this limit exists.

5. (10 points) Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Show that

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\} \leq \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

You can take the relation between the upper and lower sums of f with respect to a partition and a refinement of that partition for granted. (Recall that P' is a refinement of P if P' is obtained from P by introducing more partition points.)

6. (10 points) Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 5 + e^{3x} - \sin(\cos 4x) & \text{if } x \neq \frac{1}{n} \text{ for any } n \in \mathbb{N} \\ 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Show that g is integrable on $[0, 1]$. You cannot use the Riemann-Lebesgue Theorem from the last day of class. Hint: For any positive c, there are only finitely many n such that $c < \frac{1}{n}$.

7. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 + 5e^{-1/x} & x > 0 \\ 0 & x \leq 0, \end{cases}$$

which is integrable on any closed interval, and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_0^{x^3} f(t)\sin(t)\,dt.$$

Show that F is continuously differentiable on $\mathbb{R}$ but not twice differentiable on $\mathbb{R}$.

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