MATH 320-1: Midterm 1 Solutions
Northwestern University, Fall 2023

1. Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{R}$ which has a supremum but not an infimum.

(b) An unbounded sequence in $\mathbb{R}$ with at least two convergent subsequences.

(c) A Cauchy sequence all of whose terms are irrational.

Solution. (a) The interval $(-\infty, 0]$ has supremum 0 but no infimum since it is not bounded below.

(b) The sequence where $x_{2n} = n$ and $x_{2n+1} = (-1)^n$ works. The even-indexed terms make this sequence unbounded, but for $n = 2k$ even the subsequence $x_{2(2k)} = 1$ converges, as does the sequence $x_{2(2k+1)+1} = -1$ for $n = 2k+1$ odd.

(c) The sequence $x_n = \frac{\sqrt{2}}{n}$ is Cauchy since it converges, but all terms are irrational. □

2. Show that the supremum of the following set is 4.

$$A = \left\{ \frac{4n^2 - 6n}{n^2 - n + 1} \;\middle|\; n \in \mathbb{N} \right\}$$

Proof. For any $n \in \mathbb{N}$, we have

$$\frac{4n^2 - 6n}{n^2 - n + 1} \leq \frac{4n^2 - 4n}{n^2 - n + 1} \leq \frac{4n^2 - 4n}{n^2 - n} = 4,$$

so 4 is an upper bound of the given set. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\epsilon}{12}$$

and $N \geq 2$. Then

$$\begin{aligned} 4 - \frac{4N^2 - 6N}{N^2 - N + 1} &= \frac{-4N + 4 + 6N}{N^2 - N + 1} \\ &= \frac{2N + 4}{N^2 - N + 1} \\ &\leq \frac{2N + 4N}{N^2 - N} \\ &\leq \frac{6N}{N^2 - \frac{1}{2}N^2} \\ &= \frac{6N}{\frac{1}{2}N^2} \\ &= \frac{12}{N} \\ &< \epsilon, \end{aligned}$$

where $N \geq 2$ is needed to ensure that $\frac{1}{2}N^2 \geq N$ in the fourth step. This gives

$$4 - \epsilon < \frac{4N^2 - 6N}{N^2 - N + 1}$$

for this particular choice of N, which shows that 4 is indeed the supremum of the given set. □

3. Suppose the sequence (x_n) converges to $1/2$. Show, using the definition of convergence, that the sequence $(3/x_n^2)$ converges to $12 = 3/(1/2)^2$.

Proof. Let $\epsilon > 0$. Since $x_n \to \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that

$$|x_n - \tfrac{1}{2}| < \min\{\tfrac{1}{4}, \tfrac{\epsilon}{12\cdot 16(\frac{3}{4}+\frac{1}{2})}\} \text{ for } n \geq N.$$

This in particular implies that

$$x_n \in (\tfrac{1}{2} - \tfrac{1}{4}, \tfrac{1}{2} + \tfrac{1}{4}) \text{ for } n \geq N, \text{ so } \tfrac{1}{4} < |x_n| < \tfrac{3}{4}$$

for such n. Thus for $n \geq N$, we have

$$\begin{aligned}
\left|\frac{3}{x_n^2} - \frac{3}{(1/2)^2}\right| &= \frac{3|x_n - \frac{1}{2}||x_n + \frac{1}{2}|}{\frac{1}{4}x_n^2} \\
&< \frac{12|x_n - \frac{1}{2}||x_n + \frac{1}{2}|}{(\frac{1}{4})^2} \\
&\leq 12 \cdot 16|x_n - \tfrac{1}{2}|(|x_n| + \tfrac{1}{2}) \\
&\leq 12 \cdot 16|x_n - \tfrac{1}{2}|(\tfrac{3}{4} + \tfrac{1}{2}) \\
&< 12 \cdot 16(\tfrac{3}{4} + \tfrac{1}{2})\tfrac{\epsilon}{12\cdot 16(\frac{3}{4}+\frac{1}{2})} \\
&= \epsilon,
\end{aligned}$$

so $(3/x_n^2) \to 3/(1/2)^2$. □

4. As a consequence of a problem on the homework, the sequence (x_n) defined by $x_n = \frac{\pi^n}{n!}$ converges to 0. Using the fact that

$$x_{n+1} = \tfrac{\pi}{n+1}x_n \text{ for } n \geq 1,$$

give an alternative proof that $x_n \to 0$ which does not directly use the definition of convergence. Hint: Which of b_4 and b_5 is larger? Which of b_5 and b_6 is larger? What about b_6 and b_7?

Proof. (Note: I have no idea where b_4, b_5, b_6, b_7 in the hint came from, those were supposed to be x_4, x_5, x_6, x_7.) For $n \geq 3$, $n + 1 > \pi$ so $\frac{\pi}{n+1} < 1$. Hence for $n \geq 3$, we have

$$x_{n+1} = \tfrac{\pi}{n+1}x_n < x_n,$$

so (x_n) is decreasing for $n \geq 3$. Since all x_n are positive, they are bounded below by 0, so the sequence (x_n) converges by the monotone convergence theorem. (Technically the full sequence is not monotone, but being monotone starting at $n = 3$ is good enough since the first few terms will not affect convergence.) If we denote the limit of (x_n) by L, then taking limits in

$$x_{n+1} = \tfrac{\pi}{n+1}x_n$$

gives $L = 0 \cdot L$, so $L = 0$. Hence $x_n = \frac{\pi^n}{n!}$ converges to 0 as claimed. □

5. Suppose (x_n) and (z_n) are convergent sequences and that (y_n) is a sequence such that

$$x_n \leq y_n \leq z_n \text{ for } n \geq 10.$$

Show that (y_n) has a convergent subsequence. Careful: We are not assuming that (x_n) and (z_n) converge to the same thing, so no squeeze theorem applies.

Proof. Since (x_n) and (z_n) are convergent, each is bounded, so there exist $M, P > 0$ such that

$$|x_n| \leq M, \text{ or } -M \leq x_n \leq M \text{ for all } n$$

and

$$|z_n| \leq P, \text{ or } -P \leq z_n \leq P \text{ for all } n.$$

Then for $n \geq 10$ we have

$$-M \leq x_n \leq y_n \leq z_n \leq P,$$

so the y_n's are bounded for $n \geq 10$. (If you want to be concrete, $|y_n| \leq \max\{M, P\}$ for $n \geq 10$.) Since the subsequence of y_n's starting at $n = 10$ is bounded, it has a convergent subsequence by the Bolzano-Weierstrass theorem, which is then also a convergent subsequence of the full sequence (y_n). (Or, you can take the bound on the y_n's starting at $n = 10$ and make it larger if need be to get a bound on all of (y_n) and apply Bolzano-Weierstrass to the whole thing.) $\square$