MATH 320-1: Midterm 2 Solutions
Northwestern University, Fall 2023

1. Give an example of each of the following. You do not have to justify your answer.
(a) An unbounded function $f : \mathbb{R} \to \mathbb{R}$ which is continuous only at 0.
(b) A function on $(2, 3)$ which is continuous but not uniformly continuous.
(c) A differentiable function on $(0, \infty)$ with unbounded derivative.

Proof. (a) The function defined by $f(x) = x$ for x rational and $f(x) = -x$ for x irrational works. This is unbounded, and is continuous only at 0 since this is the only point where x and $-x$ agree.

(b) The function $f(x) = \frac{1}{x-2}$ is continuous since the denominator is continuous and never zero, but is not uniformly continuous since it cannot be extended to the endpoints of $(2, 3)$ in order to remain continuous.

(c) The function $f(x) = \sqrt{x}$ works since this is differentiable on $(0, \infty)$ and $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded. $\square$

2. Using the ϵ-δ definition, show that

$$\lim_{x \to 2} (\sqrt{x} + x^2 + \pi) = \sqrt{2} + 4 + \pi.$$

(Consider the function of which we are taking the limit here as having domain $(0, \infty)$ in order to ensure that $\sqrt{x}$ actually exists.) You'll likely need $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a - b|}$ at some point.

Proof. Let $\epsilon > 0$. If $|x - 2| < 1$, then $|x| - 2 < 1$, so $|x| < 3$. Set

$$\delta = \min\{1, \tfrac{\epsilon}{10}, \tfrac{\epsilon^2}{4}\} > 0.$$

Then if $0 < |x - 2| < \delta$, we have $|x - 2| < 1$ so that $|x| < 3$, and

$$\begin{aligned}
|(\sqrt{x} + x^2 + \pi) - (\sqrt{2} + 4 + \pi)| &= |(\sqrt{x} - \sqrt{2}) + (x^2 - 4)| \\
&\le |\sqrt{x} - \sqrt{2}| + |x - 2||x + 2| \\
&\le \sqrt{|x - 2|} + |x - 2|(|x| + 2) \\
&\le \sqrt{|x - 2|} + 5|x - 2| \\
&< \sqrt{\delta} + 5\delta \\
&\le \sqrt{\tfrac{\epsilon^2}{4}} + 5(\tfrac{\epsilon}{10}) \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $\lim_{x \to 2} (\sqrt{x} + x^2 + \pi) = \sqrt{2} + 4 + \pi$ as claimed. $\square$

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$A = \{x \in \mathbb{R} \mid f(x) < x\}$$

is bounded and nonempty, so that $c = \sup A$ exists. Show that if $f(c) > c$, then there exists $\delta > 0$ such that $f(x) > x$ for $x \in (c - \delta, c + \delta)$, and use this to conclude that we must in fact have $f(c) \le c$.

Proof. The function $g(x) = f(x) - x$ is continuous. If $f(c) > c$, then $g(c) = f(c) - c > 0$, so there exists $\delta > 0$ such that

$$g(x) > 0 \text{ for } x \in (c-\delta, c+\delta).$$

To be concrete, by continuity there exists $\delta > 0$ such that

$$|g(x) - g(c)| < g(c) \text{ when } |x - c| < \delta,$$

and this gives

$$-g(c) < g(x) - g(c) < g(c), \text{ and hence } 0 < g(x)$$

on $(c-\delta, c+\delta)$. Thus we have $f(x) > x$ on $(c-\delta, c+\delta)$.

By properties of supremums, there exists $p \in A$ such that $c - \delta < p \leq c$, which implies that $p \in (c-\delta, c+\delta)$ and hence that $f(p) > p$. This contradicts $p \in A$, so $f(c) > c$ is not possible and we conclude that $f(c) \leq c$. (The setup of this problem is similar in spirit to the proof of the intermediate value theorem, where the overall goal is to use continuity in order to turn information about the behavior of f at one point, c in this case, into information about the behavior of f at points nearby.) $\square$

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x\sqrt{x+1} & \text{if } x \geq 0 \\ x^2 + x & \text{if } x \text{ is rational and negative} \\ -x^2 + x & \text{if } x \text{ is irrational and negative} \end{cases}.$$

Show that f is differentiable at 0 and not continuous at -1.

Proof. Let (y_n) be a sequence of negative irrationals converging to -1. Then

$$f(y_n) = -y_n^2 + y_n \text{ converges to } -(-1)^2 + (-1) = -2,$$

which is not the value of $f(-1) = (-1)^2 - 1 = 0$. Thus $f(y_n)$ does not converge to $f(-1)$, so f is not continuous at -1.

For differentiability at zero, we determine

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

We have

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} \sqrt{x+1} & \text{if } x \geq 0 \\ x + 1 & \text{if } x \text{ is rational and negative} \\ -x + 1 & \text{if } x \text{ is irrational and negative.} \end{cases}$$

Thus given $\epsilon > 0$, for $\delta = \min\{\epsilon, \epsilon^2\} > 0$ we have that if $0 < |x - 0| < \delta$, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 1 \right| = \begin{cases} |\sqrt{x+1} - 1| & \text{if } x \geq 0 \\ |x| & \text{if } x \text{ is rational and negative} \\ |-x| & \text{if } x \text{ is irrational and negative.} \end{cases}$$

$$\leq \begin{cases} \sqrt{|x|} & \text{if } x \geq 0 \\ |x| & \text{if } x \text{ is rational and negative} \\ |-x| & \text{if } x \text{ is irrational and negative.} \end{cases}$$

$$
\begin{aligned}
&< \begin{cases} \sqrt{\delta} & \text{if } x \geq 0 \\ \delta & \text{if } x \text{ is rational and negative} \\ \delta & \text{if } x \text{ is irrational and negative.} \end{cases} \\
&\leq \epsilon.
\end{aligned}
$$

Hence the desired limit has value 1, so in particular it exists and thus f is differentiable at 0. $\square$

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f(x)f'(x) = 0$ for all $x \in \mathbb{R}$. Show that f is a constant function. Hint: For any $x, a \in \mathbb{R}$, consider $f(x)^2 - f(a)^2$.

Proof. For any $x \neq a$, by the mean value theorem (applied to the function $g(x) = f(x)^2$) there exists c such that

$$f(x)^2 - f(a)^2 = 2f(c)f'(c)(x - a),$$

which by the assumption on the function gives $f(x)^2 - f(a)^2 = 0$ for all $x, a \in \mathbb{R}$. Thus $g(x) = f(x)^2$ is constant, say $f(x)^2 = C$ for all x, in which case $f(x) = \pm C$ for all x. But if there existed points which gave both $-C$ and C as values, f could not be continuous (unless $C = -C$, meaning $C = 0$, in which case $f(x) = 0$ for all x and f is constant), which it must be since it is differentiable. Thus $f(x)$ equals the same constant (either C always or $-C$ always) for all x. $\square$