Math 320-3: Final Exam
Northwestern University, Spring 2020

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{R}$ with empty interior whose boundary is $[0,1]$.

(b) A function $f(x,y)$ not continuous at $\mathbf{0}$ for which the single variable functions obtained by holding x or y constant at 0 are continuous.

(c) A non-constant C^1 function $f : \mathbb{R}^3 \to \mathbb{R}^3$ to which the Inverse Function Theorem does not apply.

(d) A non-constant function $f : [0,3] \to \mathbb{R}$ whose graph has Jordan measure zero in $\mathbb{R}^2$.

(e) A non-constant function $f : \mathbb{R}^2 \to \mathbb{R}$ for which $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \int_0^1 f(x,y)\,dy\,dx$.

Problem	Score
1	
2	
3	
4	
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7	
Total	

2. (10 points) Suppose $I \subseteq \mathbb{R}$ is connected. Show that I is an interval. (Take the definition of an interval to be a set $I \subseteq \mathbb{R}$ such that if $x < y < z$ and $x, z \in I$, then $y \in I$.)

3. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|^\alpha$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ where $\alpha > 1$. Show that f is constant. Hint: This inequality implies that f is differentiable, why?

4. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 and let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$. Take γ to be the path from $\mathbf{a}$ to $\mathbf{x}$ defined by

$$\gamma(t) = t^2\mathbf{x} + (1 - t^2)\mathbf{a},\ 0 \leq t \leq 1.$$

If the norm of Df is bounded by $\frac{1}{2}$ at any point on γ, show that $\|f(\mathbf{x}) - f(\mathbf{a})\| \leq \|\mathbf{x} - \mathbf{a}\|$.

5. (10 points) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} e^{xy} & x < y^2 \\ 30 & x = y^2 \\ y \sin(e^x) & x > y^2. \end{cases}$$

Show that f is integrable on the square $[-2, 2] \times [-2, 2]$.

6. (10 points) Define $f : [0,1] \times [-1,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{y}{x^3} & x > 0 \text{ and } -x < y < x \\ 0 & \text{otherwise} \end{cases}$$

One of the iterated integrals $\int_0^1 \int_{-1}^1 f(x,y)\,dy\,dx$, $\int_{-1}^1 \int_0^1 f(x,y)\,dx\,dy$ exists and the other does not; determine which is which.

7. (10 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is integrable over the unit ball $B_1(\mathbf{0})$ and that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, one-to-one, and has $D\phi$ invertible at all points. Suppose further that the image of $B_1(\mathbf{0})$ under ϕ is $B_1(\mathbf{0})$ itself, that $f \circ \phi = f$ on $B_1(\mathbf{0})$, and that $\det D\phi = \pm 2$ at all points of $B_1(\mathbf{0})$. Show that

$$\int_{B_1(\mathbf{0})} f(\mathbf{x})\, d\mathbf{x} = 0.$$