Math 320-3: Midterm 1 Solutions
Northwestern University, Spring 2020

1. Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{R}$ whose boundary is all of $\mathbb{R}$.

(b) A function $f(x,y)$ such that $f_x(0,0)$ does not exist but $f_y(0,0)$ does.

(c) A differentiable function $f(x,y)$ such that f_x is not continuous at $(0,0)$.

(d) A differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$D(f \circ g)(x,y) = \begin{bmatrix} 4xy + x^2 & 2xy + x^2 \end{bmatrix}$$

where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is the function $g(x,y) = (2x+y, x+y)$. (Hint: You can determine $Df(x,y)$ explicitly from the given information. Recall that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$.)

Solution. (a) The boundary of $\mathbb{Q} \subseteq \mathbb{R}$ is $\mathbb{R}$: any interval around any real number includes both elements of $\mathbb{Q}$ and elements of $\mathbb{Q}^c$ since both the rationals and irrationals are dense in $\mathbb{R}$.

(b) Take the function defined by $f(0,y) = 1$ for $y \neq 0$ and $f(x,y) = 0$ everywhere else, including at the origin. Then the single-variable function $f(x,0)$ equals the constant zero, so its derivative—which is $f_x(0,0)$—exists and equals zero. But $f(0,y)$ is 1 at all $y \neq 0$ and 0 as $y = 0$, so it is not continuous and hence not differentiable with respect to y, meaning $f_y(0,0)$ does not exist.

(c) Take the function defined by $f(x,y) = (x^2+y^2)\sin(1/\sqrt{x^2+y^2})$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. (This is a two-dimensional analog of $f(x) = x^2 \sin(1/x)$, which we used in the fall as an example of a differentiable function with discontinuous derivative.) Since

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{bmatrix} 0 & 0 \end{bmatrix}\mathbf{h}}{\|\mathbf{h}\|} = \lim_{(h,k)\to(0,0)} \frac{(h^2+k^2)\sin(1/\sqrt{h^2+k^2})}{\sqrt{h^2+k^2}} = 0$$

(bound the sine part and use the squeeze theorem), f is differentiable at $\mathbf{0}$ with $f_x(0,0) = 0$. The value of f_x elsewhere is obtained using the product and chain rules: for $(x,y) \neq (0,0)$:

$$f_x(x,y) = 2x\sin(1/\sqrt{x^2+y^2}) - (x/\sqrt{x^2+y^2})\cos(1/\sqrt{x^2+y^2}).$$

When approaching along $y = 0$ the factor in front of the cosine term becomes $x/|x|$, which does not have a limit as $x \to 0$, so the limit of $f_x(x,y)$ as $(x,y) \to (0,0)$ does not exist, and hence f_x is not continuous.

(d) No such example exists! I made a mistake when formulating this problem, and at some point turned $g \circ f$ into $f \circ g$ without checking to see if the problem still made sense. Alas, it doesn't. I'll answer the problem with what I *thought* was going to be the answer, and point out why it does not work. Everyone will get credit for this part. The function $f(x,y) = x^2 y$ is what I thought should work. Indeed, the chain rule gives:

$$\begin{bmatrix} 4xy + x^2 & 2xy + x^2 \end{bmatrix} = D(f \circ g)(x,y) = Df(g(x,y))Dg(x,y) = Df(g(x,y)) \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Multiplying both sides by the inverse of the matrix on the right gives

$$\begin{bmatrix} 4xy + x^2 & 2xy + x^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = Df(g(x,y)), \text{ so } Df(g(x,y)) = \begin{bmatrix} 2xy & x^2 \end{bmatrix}.$$

My mistake was in forgetting that this is the Jacobian matrix evaluated at $g(x,y) = (2x+y, x+y)$, not at (x,y)! If instead we knew that $Df(x,y)$ was equal to this matrix, then we would need $f_x = 2xy$ and $f_y = x^2$, so that $f(x,y) = x^2 y$ does work.

But knowing that $Df(2x+y, x+y)$ instead is equal to this matrix makes the problem impossible. Set $u = 2x + y, v = x + y$, so that $x = u - v, y = 2v - u$. Then the equality above turns into

$$Df(u,v) = \begin{bmatrix} 2(u-v)(2v-u) & (u-v)^2 \end{bmatrix}.$$

Thus we need $f_u = 2(u-v)(2v-u) = -2u^2 + 6uv - 4v^2$ and $f_v = (u-v)^2 = u^2 - 2uv + v^2$, but no such f exists: the value for f_u requires that $f(u,v)$ have a $3u^2v$ term in it, but this would give a $3u^2$ term in f_u, which is not present. Whoops! ☐

2. Let A be the region in $\mathbb{R}^2$ which lies within the the square $[-5,5]\times[-5,5]$ and outside the square $[-1,1]\times[-1,1]$. Show that A is connected. (Recall $[a,b]\times[c,d]$ denotes the rectangle consisting of points (x,y) with $a \le x \le b$ and $c \le y \le d$. A proof which relies on pictures alone is not enough.)

Proof. We show that A is path-connected, which implies it is connected. Let us first show that any point in A can be connected to the upper-left corner point $(-5,5) \in A$ via a continuous path. Let $(x,y) \in A$. If $x < 1$, so that (x,y) does not lie in the right-most strip of A with $1 \le x \le 5$, then $\gamma_1(t) = (-5t + (1-t)x, y)$ for $t \in [0,1]$ gives the horizontal line segment from (x,y) to $(-5,y)$, and $\gamma_2(t) = (-5, 5t + (1-t)y, 0 \le t \le 1$ the vertical segment from $(-5,y)$ to $(-5,5)$, so that concatenating these two gives a continuous path from (x,y) to $(-5,5)$. If (x,y) lies in the right-most strip $1 \le x \le 5$, $\gamma_1(t) = (x, 5t + (1-t)y), 0 \le t \le 1$ and $\gamma_2(t) = (-5t + (1-t)x, 5), 0 \le t \le 1$ respectively give the vertical segment from (x,y) to $(x,5)$ and horizontal segment from $(x,5)$ to $(-5,5)$, and concatenating these gives a path from (x,y) to $(-5,5)$.

Now, given any two $(x,y),(a,b) \in A$, we can take the path from (x,y) to $(-5,5)$ described above, and the path from $(-5,5)$ to (a,b) described above (or rather, the reverse of the path from (a,b) to $(-5,5)$), and concatenate these to get a path from (x,y) to (a,b). Thus A is path-connected. ☐

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x,y) = \begin{cases} \left(\frac{2x^2y-3x^4}{x^2+y^2}, 4x + y^2\right) & (x,y) \neq (0,0) \\ (0,0) & (x,y) = (0,0). \end{cases}$$

Show that f is continuous but not differentiable at $(0,0)$.

Proof. Denote the components of f by $f = (f_1, f_2)$. Since $|x| = \sqrt{x^2} \le \|(x,y)\|$ and similarly $|y| \le \|(x,y)\|$, for $(x,y) \neq (0,0)$ we have:

$$|f_1(x,y)| = \left|\frac{2x^2y - 3x^4}{x^2+y^2}\right| \le \frac{2|x|^2|y| + 3|x|^4}{x^2+y^2} \le \frac{2\|(x,y)\|^3 + 3\|(x,y)\|^4}{\|(x,y)\|^2} = 2\|(x,y)\| + 3\|(x,y)\|^2.$$

The right side approaches 0 as $(x,y) \to (0,0)$, so $f_1(x,y)$ does as well by the squeeze theorem. Also, for $(x,y) \neq (0,0)$:

$$|f_2(x,y)| = |4x + y^2| \le 4|x| + |y|^2 \le 4\|(x,y)\|^2 + \|(x,y)\|^2,$$

so $f_2(x,y)$ approaches 0 as $(x,y) \to (0,0)$ since the right side does. (Using polar coordinates is also fine.) Thus $\lim_{(x,y)\to(0,0)} f(x,y) = (0,0) = f(0,0)$, so f is continuous at $(0,0)$.

To show that f is not differentiable at $(0,0)$, we show that the first component is not differentiable at $(0,0)$. We have

$$f_1(x,0) = -3x^2 \text{ for } x \neq 0 \quad \text{and} \quad f_1(0,y) = 0,$$

so $\frac{\partial f_1}{\partial x}(0,0) = \frac{d}{dx}\big|_{x=0} - 3x^2 = 0$ and $\frac{\partial f_1}{\partial y}(0,0) = 0$. Thus $Df_1(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, so

$$\frac{f_1(\mathbf{0}+\mathbf{h}) - f_1(\mathbf{0}) - Df_1(\mathbf{0})\mathbf{h}}{\|\mathbf{h}\|} = \frac{2h^2k - 3h^4}{(h^2+k^2)^{3/2}}$$

where $\mathbf{h} = (h,k)$. In polar coordinates $h = r\cos\theta, k = r\sin\theta$, this becomes

$$2\cos^2\theta\sin\theta - 3r\cos^4\theta,$$

so we see that the limit as $r \to 0$ (or equivalently $\mathbf{h} \to \mathbf{0}$) does not exist since the value depends on what happens to θ. Thus the limit defining differentiability of f_1 at $(0,0)$ does not exist, so f_1 and hence f is not differentiable at $(0,0)$. □

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = xf(x,y)$. Show that g is differentiable at any $(x,y) \in \mathbb{R}^2$ using the definition of differentiability directly.

Proof. First, we have:

$$\frac{\partial g}{\partial x}(x,y) = f(x,y) + x\frac{\partial f}{\partial x}(x,y) \quad \text{and} \quad \frac{\partial g}{\partial y}(x,y) = x\frac{\partial f}{\partial y}(x,y).$$

Thus $Dg(x,y)$ exists and

$$Dg(x,y) = \begin{bmatrix} f(x,y) + x\frac{\partial f}{\partial x}(x,y) & x\frac{\partial f}{\partial y}(x,y) \end{bmatrix}.$$

We have:

$$\begin{aligned}
\frac{g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}) - Dg(\mathbf{x})\mathbf{h}}{\|\mathbf{h}\|} &= \frac{g(x+h,y+k) - g(x,y) - \begin{bmatrix} f(x,y) + x\frac{\partial f}{\partial x}(x,y) & x\frac{\partial f}{\partial y}(x,y) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}}{\|(h,k)\|} \\
&= \frac{(x+h)f(x+h,y+k) - xf(x,y) - f(x,y)h - xDf(x,y)\begin{bmatrix} h \\ k \end{bmatrix}}{\|(h,k)\|} \\
&= x\left(\frac{f(x+h,y+k) - f(x,y) - Df(x,y)\begin{bmatrix} h \\ k \end{bmatrix}}{\|(h,k)\|}\right) \\
&\qquad + \frac{hf(x+h,y+k) - f(x,y)h}{\|(h,k)\|}.
\end{aligned}$$

Since f is differentiable at (x,y), the expression in parentheses above has limit 0 as (h,k) approaches $(0,0)$. For the second term, using $|h| \leq \|(h,k)\|$ we have:

$$\left|\frac{hf(x+h,y+k) - f(x,y)h}{\|(h,k)\|}\right| = \frac{|h||f(x+h,y+k) - f(x,y)|}{\|(h,k)\|} \leq |f(x+h,y+k) - f(x,y)|.$$

Since f is continuous (because it is differentiable), $f(x+h,y+k) \to f(x,y)$ as $(h,k) \to (0,0)$, so this term on the right goes to 0 and hence so does the expression on the left by the squeeze theorem. Thus both terms in the limit defining differentiability of g at (x,y) go to 0 as $\mathbf{h} \to \mathbf{0}$, which shows that g is differentiable at (x,y). □

5. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}^2$ are differentiable and satisfy

$$F(x, g_1(x), g_2(x)) = \mathbf{0} \text{ for all } x \in \mathbb{R}$$

where $g(x) = (g_1(x), g_2(x))$. Write the Jacobian matrix of F at a point $(x, g_1(x), g_2(x))$ as

$$DF(x, g_1(x), g_2(x)) = \begin{bmatrix} \mathbf{b} & A \end{bmatrix}$$

where $\mathbf{b}$ is the 2×1 matrix making up the first column of $DF(x, g_1(x), g_2(x))$ and A the 2×2 matrix making up the final two columns. If A is invertible, show that

$$Dg(x) = -A^{-1}\mathbf{b}.$$

Hint: View $F(x, g_1(x), g_2(x))$ as the result of composing the function $h(x) = (x, g_1(x), g_2(x))$ with F. We did a similar problem as a Warm-Up when discussing the chain rule, only in that case g (or perhaps we called it f) was a function with only one component.

Proof. Define $h : \mathbb{R} \to \mathbb{R}^3$ by $h(x) = (x, g_1(x), g_2(x))$, which is differentiable since each component is differentiable. By assumption, we have

$$F(h(x)) = F(x, g_1(x), g_2(x)) = \mathbf{0} \text{ for all } x \in \mathbb{R}.$$

The composition $F \circ h$ is differentiable by the chain rule, and

$$D(F \circ h)(x) = DF(h(x))Dh(x) = \begin{bmatrix} \mathbf{b} & A \end{bmatrix} \begin{bmatrix} 1 \\ g_1'(x) \\ g_2'(x) \end{bmatrix}.$$

In the product on the right, the entries of $\mathbf{b}$ are multiplied by the 1 in the vector at the end, and the entries of A are multiplied by $g_1'(x)$ (first column) and $g_2'(x)$ (second column). The result of this product is thus

$$D(F \circ h)(x) = \mathbf{b} + A \begin{bmatrix} g_1'(x) \\ g_2'(x) \end{bmatrix} = \mathbf{b} + ADg(x).$$

(Note $Dg(x)$ is 2×1.)

On the other hand, $F \circ h$ is the constant 0, so its Jacobian matrix should be the zero matrix. Thus

$$\mathbf{0} = \mathbf{b} + ADg(x), \text{ and thus } ADg(x) = -\mathbf{b}.$$

Multiplying both sides on the left by A^{-1} gives $Dg(x) = -A^{-1}\mathbf{b}$ as claimed. □